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Synthesis on vortex meandering

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Far-Wake Report D 1.1.1 : Synthesis on vortex meandering


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1. Introduction

The aim of this report is to summarize the work conducted within the subtask 1.1.1 of the European research project "FAR-Wake", during the period ranging from February 2005 to May 2008. The aim of this subtask was to investigate through a number of theoretical, numerical and experimental studies the phenomenon known as "vortex meandering" or "vortex wandering", which consists of rapid and global displacements of the vortex core. This phenomenon is universally observed in wind tunnel, towing tank or flight experiments, and is the major source of uncertainties affecting the measurements. One of the main challenges of the subtask was to identify the origin of this phenomenon. Two different physical mechanisms were suspected to provide the explanation for vortex meandering, namely (a) a viscous instability affecting the vortex core, and (b) an energy growth mechanism related to the receptivity of the vortex core to external perturbations. These two mechanisms were discovered shortly before the starting date of the project, respectively by Fabre & Jacquin (2004) and Antkowiak & Brancher (2004). A number of theoretical and numerical studies were conducted during the project, in order to better
characterise these two mechanisms and to evaluate their potential effect in realistic aircraft vortices. The main results of these studies will be summarised in sections 2 and 3 of the present synthesis. A specific experimental study has also been conducted and will be reviewed in section 4. General conclusions are given in section 5.

2. Viscous instabilities

2.1. Previous knowledge

The starting point is the work of Fabre & Jacquin (2004) who reconsidered the linear stability properties of the classical $q$-vortex model defined as follows:

\begin{equation}
U = 0, \quad V = \frac{q}{r} \left(1 - e^{-r^2}\right), \quad W = W_0 + e^{-r^2}, \quad (2.1)
\end{equation}

where $(U, V, W)$ is the velocity field in the cylindrical coordinates $(r, \theta, z)$, $q$ is the swirl parameter, and $W_0$ is a uniform axial flow which can be positive (jet-like vortex), negative (wake-like vortex), or zero. They performed a temporal stability analysis and computed the amplification rates $\omega_i$ of the unstable eigenmodes as a function of the swirl number $q$, Reynolds number $Re$, axial and azimuthal wavenumbers $k$ and $m$. They found that for large enough values of the Reynolds numbers, the range of parameters leading to instability was actually much larger than previously assumed. As an example, figure 1 shows the instability domain in the $(k, q)$ plane for eigenmodes with azimuthal wavenumber $m = -1$ (which was found to have the largest instability domain). The dashed line corresponds to the inviscid results as reported by Mayer & Powell (1992). The full line displays the boundary of the instability domain for $Re = 10^4$, which is substantially

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure1.png}
\caption{Instability domain in the $(q-k)$-plane for the $q$-vortex with $m = -1$. Full line: inviscid domain according to Mayer & Powel (1992; see footnote). Dashed line: viscous result for $Re = 10^4$ (Fabre & Le Dizès, 2004). Shaded area: unstable region in the limit $Re \to \infty$.}
\end{figure}
larger. In particular, instabilities were found up to a critical swirl number \( q_c = 3.235 \), a value significantly larger than the inviscid threshold \( q_c \approx 1.5 \) as reported by Mayer & Powell (1992)\(^\dagger\). Further inspection showed that the instability domain continuously extends as the Reynold number is increased, and that it tends to occupy the whole region bounded the hyperbola of equation \( k = -m/q \) and the axes of equations \( k = 0 \) and \( q = 0 \) (shaded region in figure 1). Therefore, such instabilities are likely to exist (and to be the only unstable modes if one excludes the cooperative instabilities) in realistic trailing wake situations where the swirl and Reynolds numbers are large. Fabre & Jacquin (2004) also investigated the nature of the unstable modes and found that they are of centre-mode type, i.e. their structure is strongly localised in the vicinity of the vortex centre line. The characteristic structure of the eigenmodes is displayed in figure (2) which shows the most amplified modes with azimuthal wavenumbers \( m = -1 \) and \( m = -2 \) for \( q = 2 \) and \( \text{Re} = 10^4 \). They also showed that these centre modes are related to those existing in the swirling Poiseuille flow and described previously by Stewartson, Ng & Brown (1988).

2.2. Asymptotic analyses for general vortices (TR 1.1.1-1) 

The first work completed within the present task is a complete description of the temporal stability properties of the viscous instabilities using asymptotic analyses in the limit \( \text{Re} \to \infty \) (Le Dizès & Fabre, 2007; Fabre & Le Dizès, 2008). The work was done for a general base flow defined by its Taylor expansion around the axis, with the following form:

\[
\Omega(r) = \Omega_0 + \frac{\Omega_2 r^2}{2} + \mathcal{O}(r^4),
\]

\[
W(r) = W_0 + \frac{W_2 r^2}{2} + \frac{W_4 r^4}{24} + \mathcal{O}(r^6).
\]

\(^\dagger\) As shown recently by Heaton (2007) the threshold for inviscid instabilities is actually \( q = 2.31 \), but inviscid instabilities existing for \( 1.6 < q < 2.31 \) are extremely weak and not significant compared to the viscous ones considered here.
where $\Omega(r) = V(r)/r$ is the angular velocity field and $W(r)$ is the axial velocity field. A swirl number can still be defined in the general case as follows:

$$q = -\frac{2\Omega_0}{W_2}$$ (2.4)

The asymptotic analysis proceeded in two steps.

- First, Le Dizès & Fabre (2007) considered the situation where $Re \to \infty$ while all other parameters are fixed and of order one. They were able to demonstrate that viscous instabilities exist for all negative azimuthal wavenumbers $m$ in the whole domain bounded by the hyperbola of equation $k = -m/q$ (where $q$ is defined by 2.4 in the general case), therefore confirming the expectation of Fabre & Jacquin (2004). The analysis required to investigate the nature of the eigenmodes in four imbricated layers and matching them together. The main result is the following formula predicting the frequencies of the viscous unstable modes:

$$\omega \sim \omega_0 + Re^{-1/3}\omega_1 + Re^{-1/2}\omega_2^{(n)} + \cdots$$ (2.5)

where

$$\omega_0 = m\Omega_0 + kW_0 ,$$ (2.6)

$$\omega_1 = 3i \left( \frac{|H_0|}{4} \right)^{1/3} e^{-isgn(K_0)\pi/4} ,$$ (2.7)

$$\omega_2^{(n)} = -i \sqrt{6|K_0|} e^{isgn(K_0)\pi/4} \left( n + \frac{1}{2} \right) , \quad n = 0, 1, 2, 3, \ldots$$ (2.8)

and $K_0, H_0$ are defined as function of the base-flow parameters as follows:

$$H_0 = 2\Omega_0 k(2\Omega_0 k - mW_2) ,$$

and

$$K_0 = m\Omega_2 + kW_2 .$$

Le Dizès & Fabre (2007) also showed the existence of two other families of stable viscous centre modes with frequencies admitting a similar expansion, but with a different expression for $\omega_1$.

- In a second step, Fabre & Le Dizès (2008) investigated the neutral curves regions (i.e. the vicinity of the boundaries of the unstable domain, as shaded in figure 1), where the previous analysis fails. They had to consider successively five regions of the neutral curve. In each region, they performed a distinguished limit where $Re \to \infty$ simultaneously as some other tends to zero. This resulted in a set of reduced equations which had to be solved numerically to predict the location of the neutral curve in terms of the base-flow parameters. The analysis leads to universal results applicable to any base flow defined by its Taylor expansion at the vortex centre as specified above, except for the upper neutral curve where the dependence with respect to the base flow is more intricate. In practice, the most important result of this study is the result in the 'critical swirl number' region, which leads to the conclusion that a vortex is unstable to viscous centre modes whenever

$$|\Omega_2||W_2^2|^{-4/3} < 0.1408Re^{1/3} .$$ (2.9)

Note that when applied to the $q$-vortex model this formula leads to the expression $q_c \sim 0.1408Re^{1/3}$.

All these asymptotic results were successfully compared to numerical results for the case of the $q$-vortex model. As an illustration, we show in figure 2 the amplification rates $\omega_1$ as function of $k$ for the case $m = -1, q = 2, Re = 5 \cdot 10^5$. The thick lines correspond...
Figure 3. Amplification rate $\omega_i$ for the $q$-vortex with $m = -1$, $q = 2$, $Re = 5 \cdot 10^5$. Thick lines: numerical results from Fabre & Jacquin (2004). Thin lines: asymptotic results for lower neutral curve and upper neutral curve scalings. Dashed lines: Asymptotic results from Le Dizès & Fabre (2007).

Figure 4. Neutral curves in the $(k,q)$-plane for the $q$-vortex, for $Re = 10^4$ and $m = -1$. Full line is the numerical result from Fabre & Jacquin (2004). Other lines are asymptotic predictions in the various regions considered in Fabre & Le Dizès (2008). Dashed lines: regions (1) and (1'); Dotted line: region (2); Dash-dotted line: regions (3) and (3'). Thin line: $k = -m/q$. 
to the numerical results from Fabre & Jacquin (2004). The thin lines correspond to the asymptotic prediction with the lower and the upper neutral curves scalings, from Fabre & Le Dizès (2008). Finally, the dashed lines give the asymptotic prediction away from the neutral curves from Le Dizès & Fabre (2007) (equation 2.5 of the present paper). The figures nicely illustrates how the different approximations match together. For $k < 0.1$ and $k > 0.4$, the numerical results closely follow the results with the neutral curve scalings. On the other hand, in the central region of the plots, equation (2.5) gives a good estimation to the numerical branches. As a second illustration, Figure 4 displays the neutral curves in the case $Re = 10^4$, $m = -1$. The thick lines correspond to the numerical results from Fabre & Jacquin (2004). The other lines correspond to the asymptotic predictions in the various regions considered by Fabre & Le Dizès (2008). Note the existence of a “forbidden interval” on a portion of the upper neutral curve, where no centre-modes exist.

2.3. Spatial stability analyses (TR 1.1.1-3)

These previous temporal stability and asymptotic analyses were corroborated from the viewpoint of a spatial stability analysis for the $q$-vortex model defined above (equation 2.1). Although a spatial stability analysis is numerically more involved than a temporal one, since a non-linear eigenvalue problem has to be solved, it has the advantage of providing directly physical relevant information such as the frequency ranges of the unstable modes in terms of the other parameters of the flow, namely the Reynolds number and the swirl parameter, and in terms of the azimuthal wave number of the perturbations. It also provides the onset of absolute instability in a quite straightforward way (Parras & Fernandez-Feria 2007). In addition, a spatial stability analysis is the only appropriate one to account for non-parallel effects based on the Parabolized Stability Equations (Del Pino et al. 2008). These two works performed for this work package considered the viscous centre modes for high Reynolds numbers and large swirl numbers discussed above. Previous spatial stability analysis of Batchelor vortex (Olendraru & Sellier 2002) did not explore these viscous modes for large $q$. A very accurate numerical method was used, that was able to capture with precision these centre modes for $Re$ larger than $10^8$.

Figure 5 shows a typical set of spatial stability results for $Re = 10^4$ with $W_0 = 0$, including a comparison with the asymptotic approximation for the upper and lower neutral curves described in the previous section (Le Dizès & Fabre, 2007). The agreement between asymptotic and numerical results improves very quickly as $Re$ increases. Within the neutral curves for convective instability, depicted in Fig. 5 for $Re = 10^4$, it is found that the flow may become absolutely unstable (if $W_0 < 0$) below an absolute critical swirl $q_{ca}$, which corresponds to a saddle point in the dispersion relation for an absolute frequency $\omega_0$. For instance, Fig. 6 shows the onset of absolute instability ($q_{ca}, \omega_0$) (with an asterisk mark inside the region for convective instabilities) for $Re = 10^4$ and $W_0 = -0.75$, and for perturbation with azimuthal wavenumber $m = -1$. Figure 7 shows these results for the onset of absolute instabilities as function of $W_0$ for several values of $Re$ and for perturbations with $m = -1$, which are the first to become absolutely unstable as $q$ is decreased. The summary results for the maxima of $q_{ca}$ as functions of the Reynolds number, for $m = -1, -2, -3$, are plotted in Fig. 8. For large $Re$ they follow the same asymptotic trends $q_{ca, \text{max}} \sim c_m Re^{1/3}$ as the convective instabilities. It is worth to note that the corresponding absolute frequencies for the most unstable modes with $m = -1$ are about one order of magnitude larger than the frequencies observed in vortex meandering experiments (see section 4 below).

The second step in the spatial stability analysis performed in this subtask has been to consider non-parallel effects on the stability of the most unstable modes for Batchelor’s
vortex using Parabolized Stability Equations (PSE) (Del Pino et al. 2008). To that end we have used the original self-similar solution given by Batchelor (1964), including its axial variation, and have compared the evolution of the most unstable perturbations using PSE with the near-parallel (NP) local stability results at each axial station $z$. A significant case for $Re_1 = 10^5$ and $q_1 = 0.0325$, for the most unstable frequency corresponding to a perturbation with $m = -1$, is given in Fig. 9 (the details of the relationships between the local Reynolds number $Re$ and the local swirl parameter $q$ with $Re_1$ and $q_1$ based on the freestream axial velocity and the circulation of the vortex can be found in Del Pino et al. (2008)). The main result is that the stabilization distance of trailing vortices predicted by a local, near-parallel approximation is much larger than that predicted from the PSE, which takes full account of non-parallel effects, for the high Reynolds numbers of interest in actual trailing vortices and for perturbations with $m = -1$, which is the first mode to become unstable as the swirl parameter $q_1$ is increased.
Figure 6. Regions of instability (neutral curve, $\gamma = 0$) in the $(\omega, q)$-plane for the most unstable mode with $m = -1$ for $Re = 10^4$, and $W_0 = -0.75$. The asterisks inside the curves mark the frequency ($\omega_0$) and swirl parameter ($q_{ca}$) for the onset of absolute instability (the lines ending at the asterisks correspond to the maximum growth rate for convective instabilities).

Figure 7. $q_{ca}$ (a), and $\omega_0$ (b) as functions of $W_0$ for $m = -1$ and several values of $Re$ (as indicated).
Figure 8. $q_{ca,max}$ as function of $Re$ for the three values of $m$ considered. For large $Re$ these curves behave as $q_{ca,max} \simeq c_m Re^{1/3}$ (dashed lines), with $c_m \simeq 0.1408$ for $m = -1$, 0.1142 for $m = -2$, and 0.0858 for $m = -3$.

Figure 9. Growth rate $\gamma$ (a) and wavenumber $\alpha$ (b) as a functions of $Z = zRe_1/4$ from the PSE (lines) and from the local, near-parallel stability equations (dots) for several modes corresponding to $m = -1$, $\omega = -0.6$, $Re_1 = 10^5$, in a Batchelor vortex with $q_1 = 0.06$. 

$$q_{ca,max}$$
3. Energy growth mechanism (TR 1.1.1-2)

In this section, we summarise the theoretical work aimed to investigate the explanation (b) for the vortex meandering, i.e. the result of an energy growth mechanism related to the vortex receptivity to external perturbations. We focus here on the Lamb–Oseen vortex, i.e. a Gaussian vortex with no axial flow, which normalised angular velocity is defined by

$$\Omega(r) = \frac{1}{r^2} \left(1 - e^{-r^2}\right).$$

(3.1)

The characteristics scales used for normalisation are the vortex dispersion radius \( r_0 \) and the angular velocity at the axis \( \Omega_0 \). Since there is no axial flow the Lamb–Oseen vortex is modally stable (Fabre et al. 2006). Thus, the use of this model avoids any competition with the viscous trailing instabilities mentioned in the previous section and makes the interpretation of the results clearer.

3.1. Optimal perturbation analysis

Antkowiak & Brancher (2004, 2007) calculated the “most dangerous initial condition” or optimal perturbation through an optimisation process consisting in identifying the initial condition that maximises its energy growth for any given time. Analysing the temporal evolution of these optimal disturbances led them to identify and understand the underlying physical mechanisms of energy growth. In the case of helical perturbations \( (m = 1) \), the optimal perturbation is initially composed of a pair of left-handed spiraling vorticity sheets. These two folded vorticity layers located in the quasi-potential region of the flow are of alternate sign. Thus their respective velocity induction on the vortex core initially cancel each other. As time evolves, they progressively uncoil via an Orr (1907a,b) mechanism induced by the base flow differential rotation. Through this process, the reorganisation of the vorticity sheets promotes an increasing velocity induction in the vortex core that leads to the forcing and subsequent emergence of a Kelvin wave of the Lamb–Oseen vortex. This phenomenon was interpreted as a transient resonance mechanism (Antkowiak 2005, Pradeep & Hussain 2006): a Kelvin wave is excited by the perturbation field induced by the uncoiling of the initial vorticity spirals. For the large wavelengths characteristic of the vortex meandering, the displacement wave (referred to as the D wave, see Fabre et al. 2006) emerges.

3.2. Stochastic forcing of the vortex

Given the existence of such optimal disturbance leading to the growth of the D-wave, the point is to know if it can naturally emerge from uncontrolled perturbations as diverse as atmospheric turbulence, background noise in wind tunnel experiments or turbulence generated by the aircraft wings. Indeed, while the potential for substantial transient growth of properly defined initial perturbations certainly exists, recurrent critics against optimal perturbation analyses concern the particular structure of these disturbances. These can be quite intricate and unlikely to occur spontaneously in real conditions since there is no apparent mechanism to excite such specific perturbations. This issue was theoretically addressed by Fontane et al. (2008). The general technique is to linearise the Navier-Stokes equations of small perturbations to a particular mean flow and then to augment these linear dynamics with stochastic forcing, which is uncorrelated in time (i.e. “white noise”) and also possibly uncorrelated in space. This general maintained forcing is intended to mimic perturbations arising incessantly in real transitioning flows due to background turbulence or any kind of uncontrolled (in space and time) ambient fluctuations. The approach conducted was to understand the role played by the transiently growing disturbances found in the previous optimal perturbation analyses when
the forcing lacks the bias of any specific forcing function. In that context, the associated
dynamical equations can be thought of as a system where background noise is regarded
as an “input” and the resulting random velocity field representing the response of the
flow as the “output”. The ratio of the output energy or variance to that of the input noise
gives the energy amplification of the system (Schmid 2007). Furthermore, a treatment
equivalent to the POD provides the decomposition of both statistical input and output
fields into coherent structures ordered according to their contribution to respectively the
excitation of the flow and the variance of the response.

For all azimuthal wavenumbers investigated, energy amplification was always observed
but when \( m > 2 \), the levels reached were too small compared to those obtained from
smaller values of \( m \) to be significant. Compared to the optimal perturbation analyses,
the levels of amplification obtained were always higher since it corresponds to energy
growth resulting from a continuous noise input and not from a single initial condition.
Focusing on the physical mechanisms leading to energy growth, the scenarios leading to
the vortex excitation identified by Antkowiak & Brancher (2004, 2007) and Pradeep &
Hussain (2006) were recovered. This study confirmed that the optimal perturbations can
naturally be activated by the background noise present in uncontrolled conditions.

In the helical case and for large wavelengths, the dominant input structure extracted
from the noise is identified to be composed of the same spiraling vorticity arms than
those of the optimal perturbation, and the main coherent structure of the vortex re-
sponse consists in the D mode, see figure 10. This resonance-driven emergence of the
displacement wave under a continuous external perturbation field is therefore a potential
source of vortex meandering.

3.3. Discussion

The results obtained in the present nonmodal stability analyses show that the energy gain
for the bending modes \( m = 1 \) increases indefinitely as \( k \) tends to zero. This theoretical
lack of intrinsic axial wavelength selection at small \( k \) raises questions regarding the use
of the present results in order to predict the response of the vortex in real-life conditions.
If one wants to predict the selection of a particular wavelength, the extrapolation of
the results presented in the present study must take into account the departure of the
The largest amplifications are obtained in the limit of infinite wavelengths \( (k \to 0) \) for every azimuthal wavenumber \( m \), suggesting that the response of the vortex should be systematically quasi-two-dimensional (i.e., long-wave), and involve very large structures. But it must be kept in mind that the input structures triggering the vortex response (i.e., axial vorticity spirals for \( m = 1 \)) are localised radially further away from the vortex axis when \( k \) decreases, eventually extending to infinity for \( k \to 0 \). This singular behaviour marks the limit of validity of the unbounded flow hypothesis that is implicitly made in the present formulation. It is expected that the predictions concerning the long-wave response of the vortex will be distorted by the presence of physical boundaries at finite distance such as the side walls of the wind tunnel, or by taking into account flow features ignored in the present model such as other vortices for instance. In other terms, the present study found no intrinsic wavelength selection by the vortex and we conjecture that the process of wavelength selection is extrinsic and therefore case-dependent.

Nevertheless, the present results show that the physical mechanism of growth involved in the response of the vortex systematically favour the structures with the largest axial wavelength admissible as long as their radial extent does not exceed the limit of representativeness of the vortex flow model used here. We then expect the present results to be relevant above a critical axial wavenumber \( k_c \sim 2\pi/r_c \) corresponding to a characteristic radial extent \( r_c \) above which the vortex flow model used here significantly departs from the real-life conditions. Another way of considering this issue in wind tunnel experiments is to consider that a large part of the forcing is located at a distance \( r_c \) corresponding to the wind tunnel lateral dimension. We expect relatively large flow fluctuations at the wall that could constantly feed the forcing and thus promote a response of the vortex to forcing structures located at that distance from the vortex axis. Under such forcing conditions, the corresponding wavelength is of order of magnitude of the transverse dimension of the wind tunnel.
4. Water tunnel experiments (TR 1.1.1-4)

The goal of this work was to provide detailed experimental data about vortex meandering. For this purpose, a single trailing vortex was generated in a water channel, using a half-wing. Nine configurations were tested, involving three different free-stream velocities (of about 46.6, 67, and 90 cm/s) and three different angles of attack (α = 6°, 9°, and 12°).

4.1. Base-flow characterisation

Detailed measurements of the three-dimensional vortex velocity profiles were carried out at 11.2 chord lengths behind the wing, involving stereo-PIV and recentering of individual PIV fields before averaging. The swirl velocity in the cross-sectional plane was found to correspond closely to the profile of the VM2 vortex model proposed by Fabre & Jacquin (2004a). This model consists of an inner core or radius \( a_1 \) in almost solid-body rotation, and intermediate region extending up to an outer core radius \( a_2 \) where the flow follows a power law with exponent close to \(-\frac{1}{2}\), and an external potential region. On the other hand, the axial velocity profile was to a good approximation Gaussian with characteristic radius \( a_w \).

As an illustration, figure 11 displays the velocity laws obtained for one case, namely \( \alpha = 6° \), and \( U_\infty = 46.6 \text{m/s} \). The experimental result is displayed with symbols and the fitting with VM2 and Gaussian law are displayed with lines. In this case the Reynolds number is \( \Gamma/\nu = 8700 \), the axial flow parameter is \( W_0 = 2\pi a_1 W_{r=0}/\Gamma = 0.27 \), and the characteristic radii are \( a_1 = 0.48 \text{cm}, a_2 = 2.66 \text{cm}, \) and \( a_W = 0.63 \text{cm} \).

In the other tested configurations, the axial velocity parameter \( W_0 \) was comprised between 0.27 and 0.55. This corresponds to swirl numbers \( q = W_0^{-1} \) in the range 1.8–3.7. The Reynolds number \( Re_\Gamma = \Gamma/\nu \) based on circulation varied in the range 8700–31500, while the Reynolds number \( Re_W = W_{r=0} a_w/\nu \) based on axial flow was in the range varied in the range 490–1270.
4.2. Analysis of meandering

The first part of the meandering analysis dealt with the statistics of the vortex center positions in the measurement plane. It was found that the amplitude of vortex displacement decreases with Reynolds number, in agreement with Devenport’s Devenport et al. (1996) earlier findings. The meandering amplitude was found to be of the order of the core radius, and the principal directions of motion were identified.

In the second part, singular value decomposition (or Proper Orthogonal Decomposition - POD) analysis was carried out, using high-frequency vorticity data from PIV and dye visualisation images. After confirming that both of these inputs give similar results, long time series of dye visualizations were analyzed by POD. The most energetic perturbations were found to be displacement modes of the vortex.

As an illustration, figure 12 displays the six most energetic modes identified with this technique for the case $\alpha = 6^\circ$ and $U_\infty = 46.6m/s$. The first mode (plot a) can easily be linked to the mean (time-averaged) field. It represents an axisymmetric vortex. Modes 2 and 3, presented in figures 12(b) and 12(c), are centered on the vortex. They have an azimuthal symmetry of order 1, and represents a global lateral displacement of the vortex. The principal directions of the two modes form a $90^\circ$ angle. A linear combination of both modes is therefore sufficient to account for vortex displacements in all directions of the plane. It is possible to associate modes 2 and 3 with a so-called Kelvin wave of azimuthal wavenumber $m = 1$ (see Fabre et al., 2006). The associated energy has similar values for the two modes. The next modes appear as a combination of structures with azimuthal wavenumbers $m = 0$ and $m = 2$; their energetic levels are comparable and are much lower than for modes 2 and 3.

Spectral analysis of the projection of the time series on the displacement modes showed that they are characterized by low-frequency oscillations (at the fixed measurement position) corresponding to wavelengths in the frame of reference moving with the vortex that are two orders of magnitude larger than the vortex core radius. More precisely, the axial wavenumber corresponding to these perturbations is found to be close to $ka_1 = 0.05$.

4.3. Discussion

We now discuss the previous experimental results in the light of the theoretical work presented in the previous section. The most energetic modes have been recognised as displacement modes with azimuthal wavenumber $m = 1$. This structure is precisely the one emerging from transient growth mechanisms, as identified by both the optimal perturbation analysis and the stochastic forcing (see section §3). The similarity is striking when comparing the structure of the most energetic mode (figure 12b – c), and the theoretical result (figure 10b). Moreover, the spectral analysis revealed a low frequency corresponding to a wavenumber $ka_1 = 0.05$, which corresponds to the range where the theoretical analysis predict the largest growths. This is illustrated in figure 13 which reproduces the growth factors obtained by Antkowiak & Brancher (2004) and indicates the location of the wavenumber found in the experiment. Despite the fact that the vortex velocity profile found experimentally departs from the Gaussian law used in the theory, this agreement strongly suggests that the meandering phenomenon observed in the water channel can be explained by the transient growth of vortex perturbations, initiated by background noise in the flow or by turbulence in the wake of the wing (scenario b). It is likely that this conclusion remains valid for other experimental facilities of similar type, such as wind tunnels.

On the other hand, the present experiments did not allow to identify the presence of the centre-modes studied in §2 (scenario a). In particular, none of the most energetic structure display the characteristic shape of the centre-modes as displayed in figure 2. However,
Figure 12. Modes computed by singular value decomposition of the vorticity field time series obtained by high-speed PIV for \( \alpha = 6^\circ \) and \( F_p = 25 \) Hz (\( Re = 8700 \) and \( W_0 = 0.27 \)). The six most energetic modes are shown.
in the experiment, the Reynolds number $Re_W = W_{r=0}a_w/\nu$ remain small compared to the values allowing the existence of the centre-modes. Only the configurations with the highest free-stream velocities and angle of attack (which lead to $q = W_0^{-1} = 1.8$ and $Re_W = 1270$ are compatible with their existence.

5. Conclusion

The aim of this subtask was to investigate through a number of theoretical, numerical and experimental studies the phenomenon known as ”vortex meandering”, which consists of rapid and global displacements of the vortex core. Two different physical mechanisms were suspected to provide the explanation for vortex meandering, namely (a) a viscous instability affecting the vortex core, and (b) an energy growth mechanism related to receptivity of the vortex core to external perturbations. Both these mechanisms have made the object of a consequent numerical and analytical work.

Centre-modes (mechanism a) made the object of two studies. First, a complete description of these modes and their range of existence was provided using asymptotic methods (Le Dizès & Fabre, 2007 ; Fabre & Le Dizès, 2008). These studies were done in the general case in terms of the Taylor coefficients of the axial and azimuthal velocity laws at the vortex axis. As a consequence, the results obtained are universal and can be applied to any particular vortex base-flow observed in trailing wakes as well as in many other contexts. Secondly, the spatial stability properties of these centre-modes were investigated for the $q$-vortex model (Parras & Fernandez-Feria 2007), as well as for the Batchelor solution which incorporated nonparallel effects (Del Pino et al. 2008). Here again, the results are quite general and can be applied for vortex flows observed in other contexts.

Transient energy growth (mechanism b) has been investigated through an optimal per-
turbation analysis (Antkowiak & Brancher, 2004; 2007) and a stochastic forcing analysis (Fontane et al., 2008). Both these studies revealed that the perturbations which are most likely to be obtained as a result of initial or continuous external noise are associated with very long wavelengths and consist in displacements of the vortex core as a whole. These perturbations are selected by a resonant excitation due to noise located outside of the vortex.

An experimental study was subsequently conducted. For this purpose, a single trailing vortex was generated in a water channel, using a half-wing. Dye visualisation and stereo-PIV measurements were used to investigate the properties of vortex meandering. The data were processed using POD to identify the most energetic modes present in the vortex. The results indicate that mechanism (b) is actually present in the experiment. In particular, the most energetic modes identified by PIV display both the characteristic structure, time scale and wavelength of the perturbations selected by transient growth. The presence of centre-modes (mechanism a) was not demonstrated in the experimental results. However, the Reynolds numbers reached in the experiment are not large enough to allow their existence. They could be present for larger values of the Reynolds number and provide an additional source of energy for vortex meandering, but are not expected to be the dominant mechanism in real trailing wakes because of their relatively small amplification rates.

Finally, it should be emphasised that all the theoretical studies conducted in this subtask were done for general vortices and are potentially applicable to many other contexts in fluid mechanics involving columnar vortices, such as hurricanes, rotating pipe flows, turbulent flows, etc...

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Appendices
The appendix of this deliverable contains the following documents:

- JFM article by Le Dizès & Fabre (2007).
- JFM article by Le Fabre & Le Dizès (2008).
- JFM article by Antwowiak & Brancher (2007).
- Article submitted to JFM by Fontane, Brancher & Fabre (2008).
- JFM article by Parras & Fernandez Feria (2007).

Note: Technical report TR 1.1.1-1, 1.1.1-2 and 1.1.1-3 are not reproduced since their content is essentially similar to the publications.
Large-Reynolds-number asymptotic analysis of viscous centre modes in vortices

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This paper presents a large-Reynolds-number asymptotic analysis of viscous centre modes on an arbitrary axisymmetrical vortex with an axial jet. For any azimuthal wavenumber $m$ and axial wavenumber $k$, the frequency of these modes is given at leading order by $\omega_0 = m\Omega_0 + kW_0$ where $\Omega_0$ and $W_0$ are the angular and axial velocities of the vortex at its centre. These modes possess a multi-layer structure localized in an $O(Re^{-1/6})$ neighbourhood of the vortex. By a multiple-scale matching analysis, we demonstrate the existence of three different families of viscous centre modes whose frequency expands as $\omega(n) \sim \omega_0 + Re^{-1/3}\omega_1 + Re^{-1/2}\omega_2(n)$. One of these families is shown to have unstable eigenmodes when $H_0 = 2\Omega_0k(2k\Omega_0 - mW_2) < 0$ where $W_2$ is the second radial derivative of the axial flow in the centre. The growth rate of these modes is given at leading order by $\sigma \sim (3/2)(H_0/4)^{1/3}Re^{-1/3}$. Our results prove that any vortex with a jet (or jet with swirl) such that $\Omega_0W_2 \neq 0$ is unstable if the Reynolds number is sufficiently large. The spatial structure of the viscous centre modes is obtained and simple approximations which capture the main feature of the eigenmodes are also provided.

The theoretical predictions are compared with numerical results for the $q$-vortex model (or Batchelor vortex) for $Re \geq 10^5$. For all modes, a good agreement is demonstrated for both the frequency and the spatial structure.

1. Introduction

Vortices with or without jets are present in most fluid dynamic applications and have been the subject of fundamental research for more than a century. Until recent numerical observations, viscosity was not believed to be a serious possible destabilizing factor. In this paper, we shall prove that most vortices with an axial jet are actually unstable with respect to viscous perturbations if the Reynolds number is sufficiently large. Our goal is to provide the characteristics of such unstable viscous modes for arbitrary vortex profiles using a large-Reynolds-number asymptotic analysis.

The stability of vortices has been the subject of an enormous amount of work. Recent reviews are available in Ash & Khorrami (1995) and Rossi (2000). Several studies have been concerned with aeronautical applications, notably to understand the so-called vortex breakdown phenomenon (see for instance Leibovich 1978). For this purpose, the $q$-vortex model (also called the Batchelor vortex, although it is actually a simplification of the Batchelor (1964) non-parallel solution) has often been considered. A quite complete picture of the stability properties of this vortex is now available for Reynolds numbers up to $Re \approx 10^6$ (see Lessen & Paillet 1974;
Mayer & Powell 1992; Fabre & Jacquin 2004). Its inviscid stability properties were also considered. The numerical studies by Lessen, Singh & Paillet (1974) and Mayer & Powell (1992) first demonstrated that the \( q \)-vortex is unstable with respect to inviscid perturbations of negative azimuthal wavenumbers in a finite interval of swirl number below \( q_c \approx 1.5 \). For a long time, this value has been considered the critical value for inviscid instability, as it was also in agreement with the general criterion of Leibovich & Stewartson (1983). However, very recently, Heaton (2007) exhibited unstable inviscid centre modes for swirl numbers near \( q \approx 2 \). These modes had been initially predicted by Stewartson & Brown (1985) using asymptotic methods. But their growth rate is so small that they are not expected to be visible in any experiment.

The role of viscosity is complex. Although it is expected to have a stabilizing effect on most inviscid modes (Lessen & Paillet 1974; Stewartson 1982; Mayer & Powell 1992), it also leads to the occurrence of unstable modes of a completely different nature. A first kind of viscous modes was discovered by Khorrami (1991, 1992). These modes exist for \( m = 0 \) and \( m = 1 \) in a limited range of swirl numbers, and their amplification rates are very weak. The second kind of viscous modes exist for negative \( m \) and are called centre modes because their structure is localized within the vortex core. These modes are the subject of this paper.

Historically, viscous centre modes were first identified in the case of swirling Poiseuille flow. The first indication of such a behaviour was found in the numerical study of Cotton & Salwen (1981). Later, Stewartson, Ng & Brown (1988) (referred herein as SNB) described these modes using asymptotic methods. First, they investigated the vicinity of the neutral curves of the swirling Poiseuille flow by letting both the Reynolds number tend to infinity and the distance to the neutral curve tend to zero in a distinguished way. This lead them to reduced equations allowing computation of the frequencies of the modes in terms of a rescaled distance to the neutral curve. When considering the limiting behaviour of the solutions as the distance to the neutral curve becomes large, they found that the most amplified mode does not match to an inviscid mode but acquires a characteristic structure with strong oscillations. They were able to describe such modes using a multiple-scale analysis involving four matched regions where different approximations have to be employed. In their conclusion, they speculated that such modes generally exist in all vortex flows, such as the \( q \)-vortex model. Our paper will provide a confirmation of this.

Despite its interest, the work of SNB has been largely overlooked. However, viscous centre modes were recently re-discovered in numerical works. At first, Olendraru & Sellier (2002) found modes with a centre-mode structure while studying the absolute–convective transition of the \( q \)-vortex for \( Re = 10^4 \). Then, Fabre & Jacquin (2004) gave a complete mapping of these modes using temporal stability theory for Reynolds numbers up to \( Re \approx 10^6 \). They mapped the unstable region in the \((q, k)\)-plane and found that it tends to occupy all the region located below the hyperbola of equation \( k = -m/q \), so that the \( q \)-vortex is actually unstable for all values of the swirl number if the Reynolds number is sufficiently high. They also investigated the structure of the most unstable modes and showed that they possess the same structure as described by SNB. However, the analysis conducted by SNB for swirling Poiseuille flow cannot be used for the general case, because this flow has a uniform angular velocity.

The goal of this paper is to provide an asymptotic description of these viscous centre modes in the large-Reynolds-number limit which is valid for all vortex flows, thus generalizing the work of SNB on swirling Poiseuille flow. The viscous modes will have the same asymptotic structure composed of four different layers around the vortex centre as obtained in SNB. However, our approach differs from that of SNB
in several ways. SNB first investigated the vicinity of the neutral curves by letting both the Reynolds number tend to infinity and the distance to the neutral curve tend to zero, and then let the rescaled distance to the neutral curves tend to infinity. We proceed in a more straightforward way by considering directly the \( Re \to \infty \) limit with all the other parameters being \( O(1) \). The asymptotic analysis close to the neutral curves is performed in Fabre & Dizes (2007). Secondly, SNB reduced the perturbation equations to a system of coupled equations for the radial and azimuthal velocities. In the present work, we shall be working with the pressure. We show that it is possible to reduce the perturbation equations to a single equation for the pressure in each characteristic region of the solution, which makes matching and analysis simpler. Finally, SNB focused their analysis on the unstable viscous centre modes. We shall see that there exist two other families of damped viscous centre modes which can be described in the same framework.

The paper is organized as follows. In §2, the basic equations are provided. The asymptotic analysis is performed in §3, where the perturbation equations are solved in four different layers and matching is shown to provide relations for the eigenfrequencies. In §4, the characteristics of the eigenmodes are provided. The consequences for the stability properties of the vortex are also discussed. In §5 an application of the results to the \( q \)-vortex model is considered. The theoretical results are compared to numerical results and a good agreement is demonstrated. The final section, §6, summarizes the main results and provides a short discussion on the mathematical structure of the viscous centre modes.

2. Basic flow characteristics and perturbation equations

We consider a general axisymmetrical vortex with axial flow whose velocity field is, in cylindrical coordinates, of the form

\[
U(r) = (0, V(r), W(r)),
\]

where both the azimuthal velocity \( V(r) \) and the axial velocity \( W(r) \) depend on the radial coordinate \( r \) only. The angular velocity \( \Omega(r) \) and the axial vorticity \( \Xi(r) \) of this flow are defined by:

\[
\Omega(r) = \frac{V(r)}{r}, \quad \Xi(r) = \frac{1}{r} \frac{d(rV)}{dr}.
\]

The flow domain is assumed to contain the symmetry axis, which constitutes the vortex centre. The flow is also assumed to be unbounded. In some cases, the results will also apply to flows of finite radial extent. The swirling Poiseuille flow studied in SNB was of finite radial extend and defined by \( \Omega(r) = r \) and \( W(r) = \varepsilon(1 - r^2) \).

Time and spatial scales are assumed to be non-dimensionalized by characteristic scales of the flow. For a flow dominated by rotation, they could be based on the axial vorticity and its radial variation scale. For a flow dominated by an axial jet, they could be based on the angular vorticity and its radial variation scale. To keep the generality of the results presented here, we have chosen not to favour one type of flow, but to present the results in a general framework. As we shall see, the results will uniquely depend upon four base-flow parameters: the rotation rate and axial velocity at the axis, denoted \( \Omega_0 \) and \( W_0 \), and the second radial derivative of these fields at the axis, denoted \( \Omega_2 \) and \( W_2 \). Results will be illustrated for the \( q \)-vortex model (or
Batchelor vortex) defined by
\[ \Omega(r) = q \frac{1 - e^{-r^2}}{r^2}, \quad W(r) = e^{-r^2}, \quad (2.3a, ba) \]
where \( q \) is the so-called swirl parameter.

The time-dependence of the basic flow associated with viscous diffusion is neglected in this work. By contrast, the effect of viscosity on the perturbations is considered. This is justified \textit{a posteriori} by the fact that the time scales associated with those effects, which will be \( O(Re^{1/3}) \), remain smaller than the \( O(Re) \) viscous diffusion time scale.

Linear normal-mode perturbations are sought in the form 
\[ (U, P) = (u, v, w, p)e^{ikz + im\theta - i\omega t}, \quad (2.4) \]
where \( k \) and \( m \) are axial and azimuthal wavenumbers and \( \omega \) is the frequency. The velocity and pressure amplitudes \((u, v, w, p)\) then satisfy the linear system:
\[ \begin{align*}
    i\Phi u + 2\Omega v &= -\frac{\partial p}{\partial r} + \frac{1}{Re} \left( \Delta u - \frac{u}{r^2} + \frac{2imv}{r^2} \right), \quad (2.5a) \\
    i\Phi v + \Xi u &= -\frac{imp}{r} + \frac{1}{Re} \left( \Delta v - \frac{v}{r^2} - \frac{2imu}{r^2} \right), \quad (2.5b) \\
    i\Phi w + W'u &= -ikp + \frac{1}{Re} \Delta w, \quad (2.5c) \\
    \frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{imv}{r} + ikw &= 0, \quad (2.5d)
\end{align*} \]

where the prime denotes the derivative with respect to \( r \), \( \Delta \) represents the Laplacian operator, defined in cylindrical coordinates by
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} - k^2, \quad (2.6) \]
and
\[ \Phi(r) = -\omega + m\Omega(r) + kW(r). \quad (2.7) \]
Perturbation amplitudes are also subject to boundary conditions: they must vanish at infinity and be bounded at the origin.

3. Asymptotic analysis and matching

3.1. Overview of the analysis

In this work, we focus on perturbations which are localized at the vortex centre. At leading order, the frequency is given by \( \omega \approx m\Omega_0 + kW_0 \) such that a critical-point singularity defined by \( \Phi = 0 \) will be close to the vortex centre. Because of this singularity, it can be expected that large amplitudes will be obtained near the vortex centre, and, therefore, that viscosity will dominate the structure of the modes in this region.

Our goal is to demonstrate that there exist families of eigenmodes whose frequencies have in the limit \( Re \to \infty \) an expansion of the form
\[ \omega \sim \omega_0 + Re^{-1/3}\omega_1 + Re^{-1/2}\omega_2 + \cdots \quad (3.1) \]
with
\[ \omega_0 = m\Omega_0 + kW_0. \quad (3.2) \]
To describe these eigenmodes, we consider their structure in four layers, as sketched in figure 1. These four different regions were also obtained by SNB. We justify the necessity of this structure and the eigenfrequency expansion (3.2) as follows.

The ‘outer region’ corresponds to \( r = O(1) \). In this region, as long as it is far away from the origin, the solution is regular and inviscid at leading order. This inviscid solution however possesses an essential singularity at the origin because this location is in fact a double critical point. Viscous effects are thus felt at a non-classical distance \( r = O(Re^{-1/6}) \) from the origin in an ‘outer viscous region’. The nature of the singularity will justify the use of WKBJ expansions for the solution in this region. Such a method is convenient to describe eigenmodes with a strongly oscillating structure, as observed in the numerics (see Fabre & Jacquin 2004). Physically, the WKBJ assumption implies that the viscosity is only active through the second radial derivative term in the Laplacian, while the effect of the other terms is negligible.

The different WKBJ expansions will turn out to possess singularities at turning points whose locations will be controlled by an \( O(Re^{-1/3}) \) correction to the leading-order frequency. This frequency correction \( \omega_1 \) will be chosen in such a way that there is a (double) turning point at the origin. This hypothesis will be justified \textit{a posteriori}. We shall see that the frequency of the first viscous modes of each family satisfies this property. The most unstable viscous centre mode will turn out to be one of these first modes.

The ‘intermediate region’ defined by \( r = O(Re^{-1/4}) \) corresponds to the double-turning-point region where the viscous eigenfrequencies are discretized. Although the scaling for the radial coordinate and for the frequency correction at \( O(Re^{-1/2}) \) can be obtained by the same arguments as classical double-turning point analysis (Bender & Orszag 1978), the present analysis is more complicated due to the existence of six different solutions. Note that in this region, viscosity is active through the two first terms of the Laplacian (i.e. second radial derivative and curvature term), while the third term (second azimuthal derivative) is not present.
Finally, a singularity associated with the use of the cylindrical coordinates will still be present near the origin in the intermediate region. This will justify the need of the 'inner region'. The $O(Re^{-1/3})$ scaling for this region will be chosen such that all the terms of the Laplacian (i.e. second radial derivative, curvature term, second azimuthal derivative) are present. This will lead to a solution in terms of regular Bessel functions.

### 3.2. Outer region

In the outer region, the solution is assumed to be non-viscous at leading order. An approximation is obtained by expanding the solution in powers of $1/Re$. At leading order, we obtain

\[ u(r, Re) \sim u_0(r), \quad v(r, Re) \sim v_0(r), \quad w(r, Re) \sim w_0(r), \quad p(r, Re) \sim p_0(r). \]  
(3.3a–c)

System (2.5) then becomes (at leading order)

\[ i\Phi^{(0)}u_0 - 2\Omega v_0 = -\frac{dp_0}{dr}, \quad (3.4a) \]
\[ i\Phi^{(0)}v_0 + \Xi u_0 = -\frac{imp_0}{r}, \quad (3.4b) \]
\[ i\Phi^{(0)}w_0 + W'u_0 = -ikp_0, \quad (3.4c) \]
\[ \frac{1}{r} \frac{d(ru_0)}{dr} + \frac{imv_0}{r} + ikw_0 = 0, \quad (3.4d) \]

where $\Phi^{(0)}(r) = -\omega_0 + m\Omega(r) + kW(r)$. This system can be reduced to a single equation for the pressure $p_0$:

\[ \frac{d^2 p_0}{dr^2} + \left( 1 - \frac{\Lambda'}{\Lambda} \right) \frac{dp_0}{dr} + \left( \frac{2m}{r\Phi^{(0)}\Lambda} (\Omega'\Lambda - \Omega\Lambda') + \frac{k^2\Lambda}{(\Phi^{(0)})^2} - \frac{m^2}{r^2} - \frac{2mkW'\Omega}{r(\Phi^{(0)})^2} \right) p_0 = 0, \]
(3.5)

where $\Lambda(r) = 2\Xi(r)\Omega(r) - (\Phi^{(0)}(r))^2$.

Because $\Phi^{(0)}(0) = 0$, the origin $r = 0$ is an essential singularity of this equation. Near $r = 0$, the two solutions of (3.5) behave as

\[ p_0^\pm \sim r^{1/2} \exp \left( \pm \frac{\beta}{r} \right), \quad (3.6) \]

with

\[ \beta = 2\sqrt{-H_0}/K_0, \quad (3.7) \]

where

\[ H_0 = 2\Omega_0 k(2\Omega_0 k - mW_2), \quad K_0 = \Phi''_0 = m\Omega_2 + kW_2. \quad (3.8a, b) \]

In (3.7), we define the square root such that $-\pi/2 < \arg(\beta) \leq \pi/2$. The solution to (3.5) which vanishes at infinity is therefore the combination

\[ p \sim A_\infty^+ p_0^+ + A_\infty^- p_0^-, \quad (3.9) \]

where $A_\infty^\pm$ are $O(1)$ constants. The explicit value of $A_\infty^+/A_\infty^-$ is not necessary for the following. We shall only need to assume that it is neither infinite nor zero. Note that, in a domain of finite extent, the same analysis will apply if the outer solution can still be expressed as (3.9) near the vortex centre.
Subdominant viscous solutions are also present in the outer region. A WKBJ approximation of these solutions can be obtained for large Reynolds numbers, as shown in Le Dizès (2004).

When $r$ reaches a distance $O(Re^{-1/6})$ from the origin, correction terms in (3.3d) become of the same order as $p_0$. In addition, viscous solutions which have been neglected in the outer region become important. At this distance from the origin, we enter the outer viscous region in which a specific analysis has to be carried out to obtain the leading-order approximation.

3.3. Outer viscous region $r = O(Re^{-1/6})$

In this region, the perturbation varies on the characteristic scale $\bar{r} = Re^{1/6}r$. Upon replacing $r$ by $Re^{-1/6}\bar{r}$ in the expansion of the outer solutions near $r = 0$, we can deduce the form of the expansion at leading order. We find that each independent solution in the outer viscous region can be sought in the following ‘wave’ (or WKBJ) form:

\[
\tilde{u} \sim Re^{-1/6} \tilde{u}_1(\bar{r}) \exp \left( \frac{Re^{1/6} \bar{\phi}(\bar{r})}{6} \right), \tag{3.10a}
\]

\[
\tilde{v} \sim \tilde{v}_0(\bar{r}) \exp \left( \frac{Re^{1/6} \bar{\phi}(\bar{r})}{6} \right), \quad \tilde{w} \sim \tilde{w}_0(\bar{r}) \exp \left( \frac{Re^{1/6} \bar{\phi}(\bar{r})}{6} \right), \tag{3.10b, c}
\]

\[
\tilde{p} \sim Re^{-1/3} \tilde{p}_2(\bar{r}) \exp \left( \frac{Re^{1/6} \bar{\phi}(\bar{r})}{6} \right). \tag{3.10d}
\]

At leading order, equations (2.5a–d) provide the following relations:

\[
\tilde{u}_1 = -\frac{im}{2\Omega_0 \bar{r}} \tilde{p}_2, \quad \tilde{v}_0 = \frac{\mu}{2\Omega_0} \tilde{p}_2, \quad \tilde{w}_0 = i \frac{2k\Omega_0 - mW_2}{i\omega_1 - iK_0 \bar{r}^2/2 + \mu^2} \tilde{p}_2, \tag{3.11a–c}
\]

where

\[
\mu(\bar{r}) = \bar{\varphi}(\bar{r}). \tag{3.12}
\]

At the next order, a relation for $\mu(\bar{r})$ is obtained as:

\[
\mathcal{L}(\mu, \omega_1, \bar{r}) \equiv \mu^2 \left( \mu^2 + i\omega_1 - iK_0 \bar{r}^2/2 \right)^2 - H_0 = 0, \tag{3.13}
\]

where $H_0$ and $K_0$ have been defined in (3.8a, b). The equivalent relation in SNB is their expression (5.19).

Therefore, in this region, the problem has been reduced to an algebraic equation of order 6 for $\mu(\bar{r})$. Thus, there are six independent ‘wave’ solutions with the form (3.10), and the general solution can be sought as a linear superposition of them. Four of them can be identified, from their behaviour for large $\bar{r}$, with viscous solutions, and the two others with non-viscous solutions. We shall denote by $\mu^{(1)}$ and $\mu^{(2)} = -\bar{\mu}^{(1)}$ the roots associated with non-viscous solutions, and by $\mu^{(3)}$ and $\mu^{(4)}$ the roots associated with subdominant viscous solutions (that is such that $\text{Re}(\mu) < 0$ for large $\bar{r}$). The other roots are defined by $\mu^{(5)} = -\mu^{(3)}$ and $\mu^{(6)} = -\mu^{(4)}$. It will turn out that the perturbation pressure in the outer viscous region can be written at leading order as

\[
\tilde{p} \sim Re^{-1/3} \left[ A^{(1)} \tilde{p}_2^{(1)} e^{Re^{1/6}\phi^{(1)}} + A^{(2)} \tilde{p}_2^{(2)} e^{Re^{1/6}\phi^{(2)}} + A^{(3)} \tilde{p}_2^{(3)} e^{Re^{1/6}\phi^{(3)}} \right]. \tag{3.14}
\]

The two dominant viscous solutions associated with $\mu^{(5)}$ and $\mu^{(6)}$ cannot be part of the solution because they cannot be matched to the outer solution. The contribution from the second subdominant viscous root $\mu^{(4)}$ could have been present, but it turns out that this solution is necessarily absent because it cannot be matched correctly to any solutions in the inner regions. To simplify the analysis, we have chosen to
perform the matching procedure implicitly and to retain only solutions which will remain present at the end of the analysis.

The equation which prescribes the pressure amplitude $\hat{p}_2$ of each ‘wave’ solution is obtained at the third order. The calculation is long but does not present any difficulty, so we only give the final result, which can be written as

$$L_\mu \frac{d\hat{p}_2}{d\bar{r}} + \left(\frac{1}{2} \mu' L_{\mu\mu} + \frac{1}{2} L_{\mu\bar{r}} + L_{\omega_1 \omega_2} + H \right)\hat{p}_2 = 0,$$

(3.15)

where the functions $L_\mu, L_{\mu\mu}, L_{\mu\bar{r}}$ and $L_{\omega_1}$ denote partial derivatives of $L$ with respect to the indexes and taken at $(\mu, \omega_1, \bar{r})$, and

$$H = \frac{L_\mu}{2\bar{r}} - 2iK_0\bar{r} \left(\mu^2 + i\omega_1 - iK_0\bar{r}^2\right)\mu.$$  

(3.16)

Equation (3.15) can be integrated as

$$\hat{p}_2(\bar{r}) = \frac{1}{\sqrt{|L_\mu|}} \exp \left(-\int^{\bar{r}} \frac{\omega_2 L_{\omega_1} + H}{L_\mu} \right).$$

(3.17)

Naturally, this expression breaks down at the point where $L_\mu = 0$. Those points are the turning points of the WKBJ approximation.

If we apply the condition that a (double) turning point is present at $\bar{r} = 0$, we obtain from $L_\mu(\bar{r} = 0) = 0$,

$$3\mu^2(0) + i\omega_1 = 0,$$

(3.18)

which gives, using (3.13),

$$4\mu^6(0) = H_0$$

and therefore

$$\omega_1 = 3i \left(\frac{H_0}{4}\right)^{1/3}.$$  

(3.19)

Thus, the hypothesis of a turning point at $\bar{r} = 0$ imposes the second term in the frequency expansion (3.2). More exactly, (3.19) prescribes three possible values of $\omega_1$. As will be seen below, not all of them will provide eigenmodes. Note also that $\text{Im}(\omega_1) > 0$ (the unstable frequency) is satisfied by two values of $\omega_1$ if $H_0 < 0$ and by only one if $H_0 > 0$.

For the remainder of the analysis, we need the form of the functions $\mu(\bar{r})$ and $\hat{p}_2(\bar{r})$ for the three ‘wave’ solutions present in the expression (3.14). The analysis is simplified if we use the following rescaling:

$$\mu = \sqrt{3} \left|\frac{H_0}{4}\right|^{1/6} \lambda, \quad \bar{r} = \sqrt{\frac{6}{|K_0|}} \left|\frac{H_0}{4}\right|^{1/6} s.$$  

(3.20a, b)

Equation (3.13) becomes

$$\lambda^2(\lambda^2 - \epsilon_1 e^{i\xi} - \epsilon_2 e^{i2\xi}) = \frac{4\epsilon_1}{27},$$

(3.21)

where $\epsilon_1 = \text{sgn}(H_0), \epsilon_2 = \text{sgn}(K_0)$ and $\xi = \{0, 2\pi/3, -2\pi/3\}$. The three values of $\xi$ correspond to the three possible values of $\omega_1$ in (3.19):

$$\xi = \arg(\omega_1) - \epsilon_1 \frac{\pi}{2}.$$  

(3.22)
Equation (3.21) defines 12 possible equations. Only six of them need be studied because the branches for $K_0 < 0$ can be deduced from those with $K_0 > 0$ by the transformation $(\lambda, \xi, s) \rightarrow (\lambda^*, -\xi, s^*)$. Moreover, the eigenmodes and the eigenfrequencies for negative $K_0$ can also be obtained by the transformation $(p, \omega) \rightarrow (p^*, -\omega^*)$ from positive $K_0$. In the following, we therefore assume $K_0 > 0$, i.e. $\varepsilon_2 = 1$.

The six different cases will be denoted 1a, 2a, 3a, 1b, 2b, 3b where the number refers to the parameter $\xi$: $\{1, 2, 3\} = \{\xi = 0, \xi = 2\pi/3, \xi = -2\pi/3\}$ and the letter to the parameter $\varepsilon_1$: $\{a, b\} = \{\varepsilon_1 = 1, \varepsilon_1 = -1\}$. The graphs of the different roots $\lambda(s)$ as $s$ is varied along the real axis are provided in figure 2 for each case. Except for cases 2b and 3a, all the roots are differential functions for all real $s > 0$. For cases 2b and 3a, two branches cross at the turning point $s_c = 3^{1/4}$. In such a case, we have to be careful to choose the correct way to continue the branches. This can be checked in two ways. The first is to perform a local analysis of the vicinity of the
turning point. This was performed by SNB in a case related to our case 2b. The second way is to verify that the branches are differential functions on a contour in the complex plane that goes from 0 to $+\infty$ and avoids $s_0$ on the ‘inviscid side’ prescribed by the large-Reynolds-number asymptotic analysis of the viscous critical layer at $s_c$ (Le Dizès 2004). We have checked that it is effectively the case here.

The functions $\lambda^{(1)}(s)$ and $\lambda^{(2)}(s) = -\lambda^{(1)}(s)$ associated with non-viscous ‘waves’ are those which vanish at $s = +\infty$. We have denoted by $\lambda^{(1)}$ the root which satisfies $-\pi/2 < \arg(\lambda^{(1)}) \leq \pi/2$ near $s = +\infty$. In that way, the solution associated with $\lambda^{(1)}$ matches with the non-viscous outer solution $p_0^+$, whereas the solution associated with $\lambda^{(2)}$ matches with the non-viscous outer solution $p_0^-$. Among the two (subdominant) viscous ‘waves’ $\lambda^{(3)}$ and $\lambda^{(4)}$, we shall assume that $\lambda^{(3)}$ denotes the root which is equal to one of the non-viscous roots at $s = 0$. The two roots $\lambda^{(2)}$ and $\lambda^{(3)}$, which are given as thick lines in figure 2, will turn out to be the only relevant roots for the matching with the inner solution. The configurations which will correspond to centre modes are also indicated in figure 2.

Finally, the final form of the phase $\phi$ and the amplitude $\bar{p}_2$ of each ‘wave’ solution in (3.14) is given in terms of the function $\lambda$ and the variable $s$ as

$$\phi(\bar{r}) = \eta A(s), \quad \bar{p}_2(\bar{r}) = A(s)[B(s)]^a, \quad (3.23a, b)$$

with

$$A(s) = A_0 \left( \frac{2e^{-i\gamma_0}e^{-3i\gamma_0}}{3s^2(3s^2 - \epsilon_1 e^{i\xi} - i\lambda^2)} \right)^{1/2}, \quad (3.24b)$$

$$B(s) = B_0 \exp \left( -\int_{\epsilon}^{s} \frac{2\sqrt{3}e^{i\gamma_0}\lambda}{3\lambda^2 - \epsilon_1 e^{i\xi} - i\lambda^2} \right), \quad (3.24c)$$

where $A_0$ and $B_0$ are normalization constants, and $\alpha$ and $\eta$ are given by

$$\eta = 3\sqrt{\frac{2}{|K_0|}} \left| \frac{H_0}{4} \right|^{1/3}, \quad \alpha = -i \frac{\omega_2 e^{i\gamma_0}}{\sqrt{6|K_0|}}. \quad (3.25a, b)$$

We first consider the behaviour of expression (3.14) for large $\bar{r}$ and the matching with the outer solution (3.9). The viscous root $\phi^{(3)}$ diverges to $-\infty$ for large $\bar{r}$, so it becomes negligible, as expected. By contrast, the function $\phi^{(1)}$ (and thus also $\phi^{(2)} = -\phi^{(1)}$) converges toward a finite value $\phi^{(1)} = \eta A^{(1)}$ with

$$A^{(1)}(s) = \int_{0}^{s} \lambda^{(1)}(s) \, ds. \quad (3.26)$$

The value of $A^{(1)}$ is reported in table 1. Note that $\text{Re}(A^{(1)}(s)) > 0$ and therefore that $\text{Re}(\phi^{(1)})$ is positive. If the behaviour of $\bar{p}_2^{(1)}$ and $\bar{p}_2^{(2)}$ for large $\bar{r}$ is written as $p_2^{(1)}(\bar{r}) \sim \bar{C}_0^{(1)} \bar{r}^{1/2}$ and $p_2^{(1)}(\bar{r}) \sim \bar{C}_0^{(2)} \bar{r}^{1/2}$, the matching between the outer region and the outer viscous solution is immediately found to require

$$A^{(1)}_\infty = \bar{A}_\infty^{(1)} \bar{C}_0^{(1)} \text{Re}^{1/12} \exp(\text{Re}^{1/6} \phi^{(1)}), \quad (3.27a)$$

$$A^{(2)}_\infty = \bar{A}_\infty^{(2)} \bar{C}_0^{(2)} \text{Re}^{1/12} \exp(-\text{Re}^{1/6} \phi^{(1)}). \quad (3.27b)$$
As the constants \( A_x^+, A_x^-, \tilde{C}_\infty^{(1)} \) and \( \tilde{C}_\infty^{(2)} \) are a priori all \( O(1) \), the relation between \( \tilde{A}^{(1)} \) and \( \tilde{A}^{(2)} \) can be written as follows:

\[
\frac{\tilde{A}^{(1)}}{\tilde{A}^{(2)}} = C_0 \exp(-2\text{Re}^{1/6}\phi^{(1)}),
\]

where \( C_0 \) is an \( O(1) \) constant. The positive sign of \( \text{Re}(\phi^{(1)}) \) implies that \( \tilde{A}^{(1)} \) is exponentially small compared to \( \tilde{A}^{(2)} \). Therefore, the contribution from the first "wave" can in practice be removed from the solution (3.14).

Finally, for the matching with the inner viscous region, we need the behaviour near the origin of the phase \( \phi \) and the amplitude \( \tilde{p}_2 \) of each of the two remaining contributions in (3.14). The phase satisfies

\[
\phi = \left| \frac{H_0}{4} \right|^{1/6} e^{i\gamma_0 \bar{r}} + \sqrt{\frac{2|K_0|}{3}} e^{i\gamma_1 \bar{r}^2/4} + \cdots.
\]

The angles \( \gamma_0 \) and \( \gamma_1 \) are obtained from the behaviour of \( \lambda \) near the origin:

\[
\gamma_0 = \arg(\lambda(0)), \quad \gamma_1 = \arg\left(\frac{d\lambda}{ds}(0)\right).
\]

The values of the constants \( \gamma_0 \) and \( \gamma_1 \) for the roots \( \lambda^{(2)} \) and \( \lambda^{(3)} \) are reported in Table 1 for the different cases (see also figure 2).

The functions \( A \) and \( B \) appearing in the amplitude \( \tilde{p}_2 \) expand near \( s = 0 \) as

\[
A(s) \sim A_0/s, \quad B(s) \sim B_0 s.
\]

A choice of the constants \( A_0 \) and \( B_0 \) can be made such that near \( \bar{r} = 0 \) \( \tilde{p}_2 \) satisfies:

\[
\tilde{p}_2 \sim \bar{r}^{-1+\alpha}.
\]

Collecting all these results and considering table 1, we see that the behaviour of the solution at the origin can finally be written in one of the two following forms depending on the case considered:

(i) for cases 1a, 2a, 3b,

\[
\tilde{p} \sim R^{-1/3} \bar{r}^{-1+\alpha} \left[ \tilde{A}^{(2)} \exp\left(R^{1/6}(-\mu_0\bar{r} + G_0\bar{r}^2/4)\right) \right] + \tilde{A}^{(3)} \exp\left(R^{1/6}(-\mu_0\bar{r} - G_0\bar{r}^2/4)\right);
\]

(ii) for cases 1b, 2b, 3a:

\[
\tilde{p} \sim R^{-1/3} \bar{r}^{-1+\alpha} \left[ \tilde{A}^{(2)} \exp\left(R^{1/6}(-\mu_0\bar{r} - G_0\bar{r}^2/4)\right) \right] + \tilde{A}^{(3)} \exp\left(R^{1/6}(\mu_0\bar{r} - G_0\bar{r}^2/4)\right).
\]

**Table 1.** Value of \( \gamma_0 \) and \( \gamma_1 \) defined by (3.30a, b) for the two roots \( \lambda^{(2)} \) and \( \lambda^{(3)} \) needed for the matching. The value of \( \Lambda_\infty^{(1)} \), defined in (3.26) is also given for \( \lambda^{(1)} \). (\( \gamma_2 = \text{sgn}(K_0) = 1 \)).

<table>
<thead>
<tr>
<th>Case</th>
<th>( \epsilon_1 )</th>
<th>( \xi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2a</td>
<td>1</td>
<td>2\pi/3</td>
</tr>
<tr>
<td>3a</td>
<td>1</td>
<td>-2\pi/3</td>
</tr>
<tr>
<td>1b</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2b</td>
<td>-1</td>
<td>2\pi/3</td>
</tr>
<tr>
<td>3b</td>
<td>-1</td>
<td>-2\pi/3</td>
</tr>
</tbody>
</table>

\( \Lambda^{(1)} = 0.466 - i 0.466 \ 0.17 + i 0.636 \ 0.636 + i 0.17 \ 0.466 - i 0.466 \ 0.17 + i 0.636 \ 0.636 + i 0.17 \)

\( \gamma^{(2)}_0 = \pi \ -2\pi/3 \ 2\pi/3 \ \pi/2 \ -\pi/6 \ -5\pi/6 \)

\( \gamma^{(2)}_1 = \pi/4 \ -\pi/4 \ -3\pi/4 \ -3\pi/4 \ -3\pi/4 \ -5\pi/4 \)

\( \gamma^{(3)}_0 = \pi \ -2\pi/3 \ -\pi/3 \ -\pi/2 \ 5\pi/6 \ -5\pi/6 \)

\( \gamma^{(3)}_1 = -3\pi/4 \ -3\pi/4 \ -3\pi/4 \ -3\pi/4 \ -3\pi/4 \ -5\pi/4 \)
where

$$G_0 = \left[ \frac{2|K_0|}{3} \right]^{1/2} e^{i\pi/4}. \quad (3.35)$$

### 3.4. Inner region $r = O(Re^{-1/2})$

Before analysing the intermediate region where the eigenfrequency selection takes place, we first consider the very close neighbourhood of the origin where the singularity associated with the use of the cylindrical coordinates must be smoothed. In this region, the local variable is $\hat{r} = Re^{1/2}r$ and the perturbation amplitudes should be expanded as

$$\hat{u} \sim \hat{u}_0(\hat{r}), \quad \hat{v} \sim \hat{v}_0(\hat{r}), \quad \hat{w} \sim \hat{w}_0(\hat{r}), \quad (3.36a-c)$$

$$\hat{p} \sim R e^{-1/2} \hat{p}_4(\hat{r}). \quad (3.36d)$$

Inserting $(3.36a-d)$ in $(2.5)$, one obtains the single equation for $\hat{p}_4$

$$\mathcal{G}(\hat{\Delta}, \omega_1) \hat{p}_4 = 0, \quad (3.37)$$

where

$$\mathcal{G}(\hat{\Delta}, \omega_1) \equiv (\hat{\Delta} + i\omega_1)\hat{\Delta}(\hat{\Delta} + i\omega_1) - H_0, \quad (3.38)$$

and

$$\hat{\Delta} = \frac{\partial^2}{\partial \hat{r}^2} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} - \frac{m^2}{\hat{r}^2}. \quad (3.39)$$

Solutions of $(3.37)$ can be written as modified Bessel functions $K_m(\mu_0 \hat{r})$ and $I_m(\mu_0 \hat{r})$ where $\mu_0$ is a complex number satisfying

$$\mathcal{G}(\mu_0^2, \omega_1) = 0. \quad (3.40)$$

This equation is the same as $\mathcal{L}(\mu_0, \omega_1, 0) = 0$ obtained in $(3.13)$. The complex number $\mu_0$ is therefore the value of $\mu(\hat{r})$ defined by $(3.13)$ at $\hat{r} = 0$. The values associated with the turning points are double roots. They are given by $(3.40)$, that is

$$\mu_0 = \pm \mu_0^{(1)} = \pm \sqrt{-i\omega_1/3}, \quad (3.41)$$

where $\mu_0^{(1)}$ is the value at $\hat{r} = 0$ of the non-viscous branch $\mu^{(1)}$, i.e. $\arg(\mu_0^{(1)}) = \gamma_0^{(1)}$ as given in table 1. For either of the two values defined by $(3.41)$, two other solutions are obtained as $\hat{r} I_m(\mu_0 \hat{r})$ and $\hat{r} K_m(\mu_0 \hat{r})$. As the solutions containing the functions $K_m(\hat{r})$ and $\hat{r} K_m(\hat{r})$ are singular at zero, a leading-order expression for the pressure in the inner region is finally obtained as

$$\hat{p}(\hat{r}) = \hat{A} I_m(\mu_0^{(1)} \hat{r}) + \hat{B} \mu_0^{(1)} \hat{r} I_m(\mu_0^{(1)} \hat{r}), \quad (3.42)$$

where $\hat{A}$ and $\hat{B}$ are constants. Note that, at this level, a third solution of the form $I_m(\mu_0^{(4)} \hat{r})$ with $\mu_0^{(4)} = 3i\omega_0/\omega_1$ could a priori be present in expression $(3.42)$. But it can be shown that the presence of this solution imposes in the outer viscous region both viscous solutions $\mu^{(4)}$ and $\mu^{(6)}$, and the latter is dominant and cannot be present. This justifies, a posteriori, the discarding of the solution $\mu^{(4)}$ in §3.3.

The behaviour of the inner solution for large $\hat{r}$, required for the matching with the outer regions, has the following form:

$$\hat{p} \sim \frac{1}{\sqrt{2\pi \mu_0^{(1)} \hat{r}}} \left( \hat{A} - \frac{(4m^2 + 3)\hat{B}}{8} \right) \left( \exp(\mu_0^{(1)} \hat{r}) + i(-1)^m \exp(-\mu_0^{(1)} \hat{r}) \right)$$

$$+ \sqrt{\frac{\mu_0^{(1)} \hat{r}}{2\pi}} \hat{B} \left( \exp(\mu_0^{(1)} \hat{r}) - i(-1)^m \exp(-\mu_0^{(1)} \hat{r}) \right), \quad (3.43)$$
where we have used the behaviour of the Bessel functions $I_m(z)$ and $zI'_m(z)$ for large $|z|$ with $0 < \arg(z) < \pi$.†

\[ I_m(z) \sim \frac{e^z}{\sqrt{2\pi z}} + i(-1)^m \frac{e^{-z}}{\sqrt{2\pi z}}, \quad (3.44a) \]

\[ zI'_m(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left( z - \frac{(4m^2 + 3)}{8} + \cdots \right) - i(-1)^m \frac{e^{-z}}{\sqrt{2\pi z}} \left( z + \frac{(4m^2 + 3)}{8} + \cdots \right). \quad (3.44b) \]

### 3.5. Intermediate region $r = O(Re^{-1/4})$

Now, let us compare the expression (3.43) with the behaviour at the origin of the outer viscous solution given by (3.33) or (3.34). In the first situation, corresponding to cases 1a, 2a, 3b, it is clear that the expression (3.33) contains no term that balance the terms with the form $\exp(+\mu_0(\hat{r}))$ in (3.43). Therefore, we can conclude that these cases do not lead to eigenmodes. On the other hand, in the second situation, corresponding to cases 1b, 2b, 3a, the expression (3.44) contains terms with the form both $\exp(-\mu_0(\hat{r}))$ and $\exp(+\mu_0(\hat{r}))$. Therefore, a matching of the exponential terms of (3.43) and (3.44) is possible, and modes with a structure corresponding to these cases can be constructed.

To match precisely the non-exponential terms of (3.43) and (3.44), we need to introduce an intermediate region, where the behaviour of the perturbation is on the scale $\hat{r} = Re^{1/4}r = Re^{1/12}\tilde{r}$. If we cancel the contributions which match with unbounded inner solutions or the dominant viscous solution in the outer region, the intermediate solution is found to be the sum of two independent solutions of the form

\[ \hat{u} \sim Re^{-1/12}\tilde{u}_1(\tilde{r}) \exp(Re^{1/12}\mu_0\tilde{r}), \quad (3.45a) \]

\[ \hat{v} \sim \tilde{v}_0(\tilde{r}) \exp(Re^{1/12}\mu_0\tilde{r}), \quad (3.45b) \]

\[ \hat{w} \sim \tilde{w}_0(\tilde{r}) \exp(Re^{1/12}\mu_0\tilde{r}), \quad (3.45c) \]

\[ \hat{p} \sim Re^{-1/3}\tilde{p}_4(\tilde{r}) \exp(Re^{1/12}\mu_0\tilde{r}), \quad (3.45d) \]

where $\mu_0$ takes the two values defined above $\mu_0 = \pm \mu_0^{(1)}$.

As in the previous sections, it is convenient to work with the pressure amplitude. Inserting (3.45) in (2.5), and after a few manipulations, we obtain to the same type of equation as above

\[ \mathcal{G}(\tilde{\Lambda}, \tilde{\Phi})\tilde{p}_4 = 0, \quad (3.46) \]

where the operator $\mathcal{G}(\tilde{\Lambda}, \tilde{\Phi})$ is defined in (3.38) and

\[ \tilde{\Phi} = \omega_1 + Re^{-1/6} \left( \omega_2 - K_0 \frac{\tilde{r}^2}{2} \right), \quad (3.47a) \]

\[ \tilde{\Lambda} = \mu_0^2 + Re^{-1/12} \left( 2\mu_0 \frac{\partial}{\partial \tilde{r}} + \mu_0 \frac{\tilde{r}}{\tilde{r}^2} \right) + Re^{-1/6} \left( \frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} - \frac{m^2}{\tilde{r}^2} \right). \quad (3.47b) \]

Equation (3.46) with (3.47a, b) contains all the terms up to order $Re^{-1/6}$.

The equations at the first three orders are then easily obtained. The leading order gives

\[ \mathcal{G}(\mu_0^2, \omega_1) = 0. \quad (3.48) \]

This is the same as $\mathcal{L}(\mu_0, \omega_1, 0) = 0$ obtained in (3.13).

† Here, for simplicity, we have assumed that $0 < \arg(\mu_0^{(1)}\hat{r}) < \pi$, which applies for cases 2a, 2b and 3a. For the other cases, slightly different expansions of the Bessel functions with $-\pi < \arg(\mu_0^{(1)}\hat{r}) < 0$ have to be considered, but similar results are obtained.
The next order \((Re^{-1/12})\) leads to

\[
\mathcal{G}_\Delta (\mu_0^2, \omega_1) \left( 2\mu_0 \frac{\partial}{\partial \tilde{r}} + \frac{\mu_0}{\tilde{r}} \right) \tilde{p}_4 = 0,
\]

with

\[
\mathcal{G}_\Delta (\mu_0^2, \omega_1) = (\mu_0^2 + i\omega_1) (3\mu_0^2 + i\omega_1).
\]

Expression (3.41) guarantees that \(\mathcal{G}_\Delta (\mu_0^2, \omega_1) = 0\), so (3.49) is also automatically satisfied. The equation for \(\tilde{p}_4\) is obtained at order \(Re^{-1/6}\). It can be written as

\[
\left[ \frac{1}{2} \mathcal{G}_{\Delta \Delta} (\mu_0^2, \omega_1) \left( 2\mu_0 \frac{\partial}{\partial \tilde{r}} + \frac{\mu_0}{\tilde{r}} \right)^2 + \mathcal{G}_\phi (\mu_0^2, \omega_1) \left( \omega_2 - K_0 \frac{r^2}{2} \right) \right] \tilde{p}_4 = 0,
\]

where

\[
\mathcal{G}_{\Delta \Delta} (\mu_0^2, \omega_1) = 3(\mu_0^2 + i\omega_1) = 2i\omega_1,
\]

\[
\mathcal{G}_\phi (\mu_0^2, \omega_1) = 2i\mu_0^2(\mu_0^2 + i\omega_1) = \frac{4}{9}\omega_1^2.
\]

Equation (3.51) thus reduces to

\[
\left( \frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} - \frac{1}{4\tilde{r}^2} + \frac{i\omega_2}{3} - iK_0 \frac{\tilde{r}^2}{6} \right) \tilde{p}_4 = 0.
\]

This equation can be written in the following more convenient form:

\[
\left( \frac{\partial^2}{\partial x^2} + v + 1 - \frac{x^2}{4} \right) (\sqrt{x} \tilde{p}_4) = 0,
\]

where the following notation has been introduced:

\[
x = \left[ \frac{2|K_0|}{3} \right]^{1/4} e^{i\pi/8} \tilde{r}, \quad v = \frac{\omega_2}{\sqrt{6|K_0|}} e^{-3i\pi/4} - \frac{1}{2}.
\]

We recognize (3.53) as the parabolic cylinder equation, typical of double-turning-point problems. Two independent solutions are \(D_v(x)\) and \(D_{-v-1}(-ix)\), where \(D_v\) is the parabolic cylinder function (see Bender & Orszag 1978). Therefore, the solution in the intermediate region can be written as

\[
\tilde{p} \sim (\tilde{A}^{(a)} D_v(x) + \tilde{A}^{(b)} D_{-v-1}(-ix)) x^{-1/2} \exp \left( +Re^{1/12} \mu_0^{(l)} \tilde{r} \right)
\]

\[
+ (\tilde{A}^{(c)} D_v(x) + \tilde{A}^{(d)} D_{-v-1}(-ix)) x^{-1/2} \exp \left( -Re^{1/12} \mu_0^{(l)} \tilde{r} \right),
\]

where \(\tilde{A}^{(b)}, \tilde{A}^{(c)}, \tilde{A}^{(e)}, \text{and} \tilde{A}^{(d)}\) are constants.

\subsection{3.6. Matching and frequency selection}

We first consider the matching between the intermediate and the outer viscous regions. For \(x \to \infty\), the asymptotic behaviour of the parabolic cylinder functions is as follows:

\[
D_v(x) \sim x^v e^{-x^{7/4}}.
\]

This expression is valid for \(|\arg(x)| < 3\pi/4\). Here \(\arg(x) = \pi/8\), and \(\arg(-ix) = -3\pi/8\), so it can be used to obtain the behaviour of both \(D_v(x)\) and \(D_{-v-1}(-ix)\). This allows us to match the outer and intermediate solutions for \(\tilde{r} \to \infty\). When expressed in
terms of the variable $\tilde{r}$ of the outer viscous region, we obtain
\begin{equation}
\hat{p} \sim (G_1 R e^{1/12 \tilde{r}})^{v-1/2} \exp \left( - Re^{1/6} G_1^4 \tilde{r}^2 \right) \left[ \hat{A}^{(a)} \exp(+Re^{1/6} \mu_0^{(1)} \tilde{r}) + \hat{A}^{(c)} \exp(-Re^{1/6} \mu_0^{(1)} \tilde{r}) \right] \\
+ (G_1 R e^{1/12 \tilde{r}})^{-v-3/2} (-i)^{-v-1} \exp \left( +Re^{1/6} G_1^4 \tilde{r}^2 \right) \\
\times [\hat{A}^{(b)} \exp(+Re^{1/6} \mu_0^{(1)} \tilde{r}) + \hat{A}^{(d)} \exp(-Re^{1/6} \mu_0^{(1)} \tilde{r})], \tag{3.58}
\end{equation}
where
\begin{equation}
G_1 = \left[ \frac{2|K_0|}{3} \right]^{1/8} e^{i\pi/16}. \tag{3.59}
\end{equation}

This expression needs to be matched with the behaviour of the outer viscous solution for cases 1b, 2b, 3a given by expression (3.34) (cases 1a, 2a, 3b have been ruled out and will not be considered). Clearly, the terms of amplitude $\hat{A}^{(2)}$ and $\hat{A}^{(3)}$ in (3.34) need to be matched, respectively, to the terms of amplitude $\hat{A}^{(c)}$ and $\hat{A}^{(a)}$ in (3.58). The two other terms in (3.58) have no counterpart in (3.34). Therefore, we must have $\hat{A}^{(b)} = \hat{A}^{(d)} = 0.$

The intermediate solution behaves, as $\tilde{r} \to 0$, as follows:
\begin{equation}
\hat{p} \sim \frac{\sqrt{\pi} 2^v}{\Gamma(1/2-v/2)} (G_1 \tilde{r})^{-1/2} (\hat{A}^{(a)} \exp(+Re^{1/12} \mu_0^{(1)} \tilde{r}) + \hat{A}^{(c)} \exp(-Re^{1/12} \mu_0^{(1)} \tilde{r})) \\
+ \frac{\sqrt{\pi} 2^{v+1}}{\Gamma(-v/2)} (G_1 \tilde{r})^{1/2} (\hat{A}^{(a)} \exp(+Re^{1/12} \mu_0^{(1)} \tilde{r}) + \hat{A}^{(c)} \exp(-Re^{1/12} \mu_0^{(1)} \tilde{r})). \tag{3.60}
\end{equation}

The behaviour at infinity of the inner solution has been obtained in (3.43). Written with the intermediate variable $\tilde{r} = Re^{-1/12 \tilde{r}}$, it is
\begin{equation}
\hat{p} \sim \frac{Re^{-1/24}}{\sqrt{2\pi \mu_0^{(1)} \tilde{r}}} \left( \hat{A} - \frac{(4m^2 + 3)}{8} \hat{B} \right) (\exp(Re^{1/12} \mu_0^{(1)} \tilde{r}) + i(-1)^m \exp(-Re^{1/12} \mu_0^{(1)} \tilde{r})) \\
+ Re^{1/24} \sqrt{\frac{\mu_0^{(1)} \tilde{r}}{2\pi}} \hat{B} (\exp(Re^{1/12} \mu_0^{(1)} \tilde{r}) - i(-1)^m \exp(-Re^{1/12} \mu_0^{(1)} \tilde{r})). \tag{3.61}
\end{equation}

Comparing these expressions and equating the coefficients of similar terms, we see that the matching requires the following two conditions:
\begin{equation}
\frac{1}{\Gamma(1/2-v/2)} (\hat{A}^{(a)} - i(-1)^m \hat{A}^{(c)}) = 0, \tag{3.62a}
\end{equation}
\begin{equation}
\frac{1}{\Gamma(-v/2)} (\hat{A}^{(a)} + i(-1)^m \hat{A}^{(c)}) = 0. \tag{3.62b}
\end{equation}

Clearly, these two equations cannot be satisfied simultaneously, unless either $1/2 - v/2$ or $-v/2$ is a singular point of the function $\Gamma$. The first condition is satisfied when $v$ is an odd and positive integer, whereas the second is satisfied when $v$ is an even and positive integer. Therefore, the general solution is $v = n$ with $n = 0, 1, 2, \ldots$.

This prescribes, using (3.55b), the value of the second-order frequency $\omega_2$:
\begin{equation}
\omega_2 = \sqrt{6|K_0| e^{-3i\pi/4}} (n + \frac{1}{2}) \text{ with } n = 0, 1, 2, 3, \ldots. \tag{3.63}
\end{equation}

For these values of $\omega_2$, cases 2b, 1b and 3a do correspond to eigenmodes. In the following, these modes will be denoted modes A, B and C respectively. Note that they
correspond to the configurations in figure 2 where the branches $\lambda^{(2)}$ and $\lambda^{(3)}$ are not connected with each other.

The conditions of matching also lead to relations between the constants $\hat{A}$, $\hat{B}$, $\tilde{A}^{(a)}$, $\tilde{A}^{(c)}$, $\bar{A}^{(2)}$ and $\bar{A}^{(3)}$ appearing in the expressions for the pressure in the different regions. These relations have been used to derive the expressions for the eigenmodes which are provided in the Appendix.

4. Characteristics of the viscous centre modes

If we recollect the results of the previous section and of the Appendix, we obtain both the frequencies and the spatial structure of the viscous centre modes. Both essentially depend on two parameters which are

$$H_0 = 2\Omega_0 k (2\Omega_0 k - m W_2), \quad K_0 = m \Omega_2 + k W_2.$$

Results were obtained for positive $K_0$. For negative $K_0$, frequency and spatial structure can be deduced from the case $K_0 > 0$ by applying the transformation $(\omega, p) \rightarrow (-\omega^*, p^*)$.

4.1. Eigenfrequencies

According to the signs of $K_0$ and $H_0$, the spectrum associated with the viscous centre modes has one of the typical forms shown in figure 3. We have identified three types of centre modes (modes A, B and C) corresponding to cases 2b, 1b and 3a. For all these modes, the eigenfrequencies expand as

$$\omega \sim \omega_0 + \text{Re}^{-1/3} \omega_1 + \text{Re}^{-1/2} \omega_2^{(n)} + \cdots$$

(4.1)

where

$$\omega_0 = m \Omega_0 + k W_0,$$

(4.2a)

$$\omega_2^{(n)} = -i \sqrt{6}|K_0| e^{i \text{sgn}(K_0) \pi/4} (n + \frac{1}{2}), \quad n = 0, 1, 2, 3, \ldots$$

(4.2b)
Viscous centre modes

If $H_0 < 0$, $\omega_1$ can take two different values:

$$
\omega_1 = 3i \left( \frac{|H_0|}{4} \right)^{1/3} e^{-\text{sgn}(K_0) \pi/3} \quad \text{(modes A),}
$$

or

$$
\omega_1 = -3i \left( \frac{|H_0|}{4} \right)^{1/3} \quad \text{(modes B).}
$$

If $H_0 > 0$, $\omega_1$ is given by

$$
\omega_1 = 3i \left( \frac{|H_0|}{4} \right)^{1/3} e^{-\text{sgn}(K_0) 2\pi/3} \quad \text{(modes C).}
$$

4.2. Spatial structure of the viscous centre modes

The eigenmodes are localized in the neighbourhood of the vortex centre but they have different approximations according to the distance from the centre as illustrated in figure 1. Approximations for the eigenmode pressure are provided in the Appendix. They are given by (A1a, b), (A3) and (A4) in the inner region, intermediate region and outer viscous region respectively. Here we shall further reduce these expressions and give approximations which capture the main features of each mode.

It is useful first to consider the outer viscous region ($r = O(Re^{1/6})$). We shall show that both modes A and C are localized in this region. In this region, the centre modes are a sum of two contributions associated with the branches $\lambda^{(2)}$ and $\lambda^{(3)}$ (see expression (A4)). The main behaviour of each contribution in (A4) is provided by the exponential term

$$
p_{\text{approx}} = \exp(Re^{1/6} \eta \Lambda(s)),
$$

where we recall that

$$
\Lambda(s) = \int_0^s \hat{\lambda}(x) \, dx,
$$

and

$$
\eta = 3 \sqrt{\frac{2}{|K_0|} \left( \frac{H_0}{4} \right)^{1/3}}, \quad s = Re^{1/6} \sqrt{\left| K_0 \right|/6} \left( \frac{4}{H_0} \right)^{1/6} r.
$$

It therefore mainly depends on the variation of $\text{Re}(\Lambda)$, i.e. on the sign of $\text{Re}(\lambda)$. Moreover, as $A^{(2)}[B^{(2)}]^{n+1/2}$ and $A^{(3)}[B^{(3)}]^{n+1/2}$ are of same order, the relative weight of each contribution also depends on the functions $\lambda^{(2)}$ and $\lambda^{(3)}$. The functions $\text{Re}(\lambda^{(2)})$ and $\text{Re}(\lambda^{(3)})$ and the functions $\text{Re}(A^{(2)})$ and $\text{Re}(A^{(3)})$ are plotted versus $s$ in figure 4(a, b) for each family of centre modes.

For modes A, $\text{Re}(A^{(3)}) < \text{Re}(A^{(2)})$ for all $s > 0$, which implies that the viscous contribution associated with $\lambda^{(3)}$ remains negligible everywhere. Note also that $\text{Re}(\lambda^{(2)})$ is positive up to $s_m^{(A)} \approx 0.931$, and then becomes negative. This implies that $|\bar{p}|$ grows exponentially up to $s_m^{(A)}$ and then decreases exponentially. This property also means that modes A are localized in the outer viscous region near the point $s_m^{(A)}$. An adequate approximation for modes A is therefore

$$
\bar{p}^{(A)} \sim A^{(2)}[B^{(2)}]^{n+1/2} \exp(Re^{1/6} \eta \Lambda^{(2)}). \quad (4.7)
$$

The functions $A^{(2)}$, $B^{(2)}$ and $\Lambda^{(2)}$ which describe these modes are plotted in figure 5. All the quantities have been normalized by their value at $s_m^{(A)}$. The function $\Lambda^{(2)}$ is regular for all $s$, but $A^{(2)}$ and $B^{(2)}$ exhibit singularities at the origin (for $A^{(2)}$) and at the turning point $s_t = 3^{1/4}$ (for both $A^{(2)}$ and $B^{(2)}$). At those points, approximation (4.7) is
therefore not valid. Close to the origin, the approximation in the intermediate region must be used while close to \( s_c = \frac{3}{4} \) a specific turning-point-like approximation must \( \textbf{a priori} \) be constructed. Such an approximation is provided in SNB.

For modes \( C \), \( \text{Re}(\Lambda^{(2)}) \) is smaller than \( \text{Re}(\Lambda^{(3)}) \) for \( s < s_d^{(C)} \approx 1.92 \) and then becomes larger. The main contribution in (A 4) is therefore associated with the viscous branch \( \lambda^{(3)} \) up to \( s = s_d^{(C)} \). Moreover, \( \text{Re}(\Lambda^{(3)}) \) is positive up \( s_m^{(C)} \approx 0.565 \). Thus, as for modes \( A \), \( |\tilde{p}| \) grows exponentially up to \( s_m^{(C)} \) and then decreases. Modes \( C \) are therefore also localized in the outer viscous region and their approximation is (up to \( s_d^{(C)} \))

\[
\tilde{p}^{(C)} \sim A^{(3)}[B^{(3)}]^{n+1/2} \exp(\text{Re}^{1/6} \eta \Lambda^{(3)}).
\]

The functions \( A^{(3)}, B^{(3)} \) and \( \Lambda^{(3)} \), which fully describe modes \( C \), are plotted in figure 6. Unlike modes \( A \), the functions \( A^{(3)} \) and \( B^{(3)} \) are regular for all positive \( s \).

For modes \( B \), \( \text{Re}(\lambda^{(3)}) < \text{Re}(\lambda^{(2)}) < 0 \) for all \( s > 0 \): modes \( B \) are therefore dominated by the non-viscous contribution but they decrease exponentially in the outer viscous region. Thus, modes \( B \) are not localized in the outer viscous region; they reach their maximum amplitude at a point closer to the centre, in the intermediate or in the inner region. The main features of modes \( B \) are not described by their approximation in the outer viscous region, but instead by the approximations obtained in the inner and intermediate regions.
A composite approximation valid in both inner and intermediate regions can be easily obtained from expressions (A 1b) and (A 3) using classical techniques (see, for instance Van Dyke 1975). We obtain the following expressions in the outer variable:

For \( n \) even,

\[
p = C \text{He}_n(\beta R^{1/4}r) \exp\left(-\frac{\beta^2 R^{1/2}r^2}{4}\right) I_m(\mu_0 R^{1/3}r),
\]

(4.9)

for \( n \) odd,

\[
p = C \text{He}_n(\beta R^{1/4}r) \exp\left(-\frac{\beta^2 R^{1/2}r^2}{4}\right)

\times \frac{(4m^2 + 3)I_m(\mu_0 R^{1/3}r) + 8\mu_0 R^{1/3}r I'_m(\mu_0 R^{1/3}r)}{r},
\]

(4.10)

where \( C \) is a normalization constant, \( \beta = \sqrt{2}e^{i\pi/8}(|K_0|/6)^{1/4} \), \( \mu_0 = -\omega_1/3 \), and \( \text{He}_n \) is the Hermite polynomial of order \( n \). The above expressions apply for modes A, B and C if we use the value of \( \omega_1 \) corresponding to each case. However, the above expressions are useful only for modes B because modes A and C are localized in the outer viscous region. For modes B, \( \mu_0 = i(|H_0|/4)^{1/6} \).

4.3. Instability characteristics

Only modes A obtained when \( H_0 < 0 \) are unstable. The frequency of the most unstable mode is given by (4.1) with (4.3) for \( \omega_1 \) and (4.2b) with \( n = 0 \) for \( \omega_2 \). Its growth rate is given up to \( O(Re^{-1/2}) \) by

\[
\sigma^{(0)} \sim Re^{-1/3} \left(\frac{|H_0|}{4}\right)^{1/3} - Re^{-1/2} \frac{\sqrt{3} |K_0|}{2}.
\]

(4.11)

For each \( m \) and fixed profile parameters, the characteristics of the most dangerous mode are obtained by maximizing \( \sigma^{(0)} \) over \( k \). The most dangerous mode is easily found to have a wavenumber and a growth rate given by

\[
k_{max} \sim \frac{mW_2}{4\Omega_0} - Re^{-1/6} \frac{3^{1/2} W_2 |mW_2|^{2/3}}{\Omega_0^{2/3}} \sqrt{\frac{\Omega_0}{m(4\Omega_0 \Omega''_0 + (W_2)^2)}},
\]

(4.12a)

\[
\sigma^{(0)}_{max} \sim Re^{-1/3} \frac{3^{1/2} |mW_2|^{2/3}}{2^{7/9}} - Re^{-1/2} \frac{\sqrt{3}}{2} \sqrt{\frac{m(4\Omega_0 \Omega''_0 + (W_2)^2)}{\Omega_0}}.
\]

(4.12b)

Note that the leading-order maximum growth rate does not depend on \( \Omega_0 \).
The condition of instability of the centre modes is $H_0 < 0$, that is

$$\Omega_0 k (2 \Omega_0 k - m W_2) < 0. \quad (4.13)$$

The marginal stability curves are provided at leading order by the condition $H_0 = 0$, which is equivalent to one of the following conditions:

$$k = 0, \quad \Omega_0 = 0, \quad 2k \Omega_0 = m W_2. \quad (4.14a-c)$$

Note however that, for $H_0 = 0$, the present asymptotic analysis breaks down. Therefore, the first-order correction cannot be obtained from (4.11). As is shown in a companion paper (Fabre & Le Dizès 2007), specific scaling must be introduced near each stability curve. The problem becomes degenerate and the three-different regions (outer viscous, intermediate, inner) merge into a single region in which the problem has in general to be solved numerically.

Note that condition (4.13) is not very restrictive. Once $\Omega_0 \neq 0$ and $W_2 \neq 0$, there exist $k$ and $m$ such that (4.13) is satisfied. This means that most non-uniform jets with swirl are unstable with respect to viscous centre modes. Note also that (4.13) is never satisfied for a vortex without a jet or a jet without swirl. The combination of swirl and jet is therefore necessary for instability although the maximum growth rate only depends (at leading order) on the jet component (see expression (4.12b)).

5. Application to the $q$-vortex model (Batchelor vortex)

In this section, the results of the previous sections are applied to the $q$-vortex model (or Batchelor vortex) (2.3a, b) and compared to numerical results, in particular those of Fabre & Jacquin (2004). For this vortex, the parameters $H_0$ and $K_0$ are given by

$$H_0 = 4 q^2 k (k + m/q), \quad K_0 = -mq - 2k. \quad (5.1a,b)$$

For positive swirl $q$ and positive wavenumber $k$, unstable centre modes are obtained for negative $m$ only. The domain of instability is defined by

$$0 < k < -m/q, \quad (5.2)$$

or

$$0 < q < -m/k. \quad (5.3)$$

The form of this domain, which is illustrated in figure 7, was suggested by the numerical results of Fabre & Jacquin (2004). In figure 7, the dash-dotted line $k = -mq/2$ corresponds to parameters for which $K_0 = 0$. This line together with the marginal stability curve $k = -m/q$ delimits four regions in which the centre-mode spectrum has one of the typical forms shown in figure 3, indicated by a letter which refers to the label in figure 3.

On the boundary of each region, either $H_0$ or $K_0$ vanishes, and therefore the estimates obtained for the centre-mode frequency do not apply. In the unstable region (indicated in grey in figure 7), the frequency of the unstable modes (modes A) are given for $n = 0, 1, 2, \ldots$ by

$$\omega^{(n)} \sim (mq + k) + \frac{3(i + \varepsilon_2 \sqrt{3})}{2} \left| \frac{k^2 q^2 + kmq}{Re} \right|^{1/3} - (i + \varepsilon_2) \sqrt{\frac{3|mq + 2k|}{Re}} (n + \frac{1}{2}), \quad (5.4)$$
Viscous centre modes

Figure 7. Domain of instability of the centre modes of negative azimuthal wavenumber $m$ for the $q$-vortex (grey region).

with $\varepsilon_2 = \text{sgn}(K_0) = -\text{sgn}(mq + 2k)$. The growth rate of the most unstable centre mode $(n = 0)$ is

$$\sigma^{(0)} \sim \frac{3}{2} \left| \frac{k^2 q^2 + kmq}{Re} \right|^{1/3} - \frac{\sqrt{3}}{2} \sqrt{\left| \frac{mq + 2k}{Re} \right|}, \quad (5.5)$$

The maximum growth rate over all $k$, for fixed $q > 0$ and $m < 0$, is

$$\sigma^{(0)}_{\text{max}} \sim 3 \left( \frac{m^2}{32Re} \right)^{1/3} - \frac{3|m(q^2 - 1)|}{4|q|Re}, \quad (5.6)$$

which is reached for

$$k_{\text{max}} \sim -\frac{m}{2q} + 2^{-7/3} |m|^{5/6} \sqrt{\frac{3}{|q^3(q^2 - 1)|}} Re^{-1/6}. \quad (5.7)$$

The leading-order expression for $k_{\text{max}}$ is in agreement with the numerical computations of Fabre & Jacquin (2004). It is also worth mentioning that the leading-order term in the maximum growth rate expression (5.6) does not depend on the swirl number $q$ and increases as $m^{2/3}$. The dependence on $q$ appears at the next order $O(Re^{-1/2})$ and is proportional to $\sqrt{q}$ (for large $q$). We therefore expect the critical swirl number to scale as $q_{\text{crit}} \propto Re^{1/3}$. This scaling is confirmed in Fabre & Le Dizès (2007) where specific analysis close to the neutral curves is performed. The numerical maximum growth rate for Reynolds numbers ranging from $10^3$ to $2 \times 10^6$ has been plotted in Fabre & Jacquin (2004) for several $m$, and $q = 2$ and $q = 3$. Formula (5.6) does not capture well the smallest Reynolds numbers but it is in good agreement with the numerics for $Re \geq 10^6$.

In figure 8(a–d), the temporal spectrum of the $q$-vortex is displayed for four sets of parameters. Both numerical results and theoretical predictions are plotted in this figure. Numerical results are obtained with a spectral code similar to the one used in Fabre & Jacquin (2004). The code has been written with Matlab® by P. Brancher and A. Antkowiak. An algebraic mapping and more than 250 Chebyshev polynomials have been used to reach an adequate resolution for $Re = 10^6$. More details concerning
Figure 8. Temporal spectra of the $q$-vortex near the centre-mode frequencies. Stars: numerical results. Circles: theoretical predictions for modes A, B and C. (a) $q = 0.4$, $k = 1.5$, $m = -1$; $Re = 10^6$; (b) $q = 4$, $k = 0.2$, $m = -1$; $Re = 10^6$; (c) $q = 1$, $k = 2$, $m = -1$; $Re = 10^6$; (d) $q = 2$, $k = 0.7$, $m = -1$; $Re = 10^5$.

The spatial structure of the eigenmodes indicated in figure 8 are compared with theoretical approximations in figures 9, 10 and 11. All the modes are normalized at
Viscous centre modes

Figure 9. Spatial structure of modes A. Solid line: $|p|$; dashed line: Re($p$). (a–c) Numerical results for modes $A_0$, $A_1$, and $A_4$ indicated in figure 8(b). (d) Theoretical spatial structure (geometrical optics approximation in the outer viscous region).

their maximum. In figure 9(a–c) the numerical results are displayed for the pressure of three different modes A corresponding to $n = 0$, $n = 1$ and $n = 4$. In figure 9(d) the theoretical pressure based on the simple geometrical optics approximation (4.6) is shown for modes A. We can see that the spatial structures of the three different modes are very close to each other and very well reproduced by the theory. The weak shift of the maximum amplitude between mode $A_0$ and mode $A_4$ can be captured by considering the amplitude corrections $A B^{n+1/2}$. However, the resulting physical optics approximation is less convenient because it breaks down near the origin and at a particular location $r_c$, indicated in figure 9(d), corresponding to the turning point $s_c$.

A similar comparison is shown for three modes C in figure 10. Again, the numerical modes are seen to be very well reproduced by the geometrical optics approximation (4.6) of modes C for the same parameters.

We have seen in the previous section that the spatial structure of modes B is of a different type because it is more localized near the vortex centre. Figure 11(a) displays the numerical mode $B_0$ indicated in figure 8(a). This mode should be compared...
Figure 10. Spatial structure of modes C. Solid line: \(|p|\); dashed line: \(\text{Re}(p)\). Plots (a–c): Numerical results for the modes \(C_0\), \(C_1\), and \(C_4\) indicated in figure 8(c). Plot (d): Theoretical spatial structure (geometrical optics approximation in the outer viscous region).

with the theoretical prediction shown in figure 11(b), which plots the composite approximation (4.9) for the first mode B. Here again, good agreement is obtained between theory and numerical results.

Finally, figure 12 displays the amplification rates of the most unstable modes in the range of Reynolds numbers \(10^3–10^8\), for the set of parameters \(m = -2, q = 2, k = 0.5\). The original method of Fabre & Jacquin (2004) was not able to compute accurately modes above \(Re \approx 10^6\). Here, the method has been modified to include a complex contour deformation procedure, as described in Fabre, Sipp & Jacquin (2006). Interestingly, the higher branches display irregular behaviour as well as branch-crossing events. Such features are characteristic of the viscous centre modes, and were systematically observed in the study of the vicinity of the neutral curves by Fabre & Le Dizès (2007). The theoretical predictions for the three first branches, obtained from (5.4) with \(n = 0, 1, 2\), are displayed with dashed lines in the figure. As can be observed, good matching between theory and numerics is only found for very large Reynolds numbers, above \(Re \approx 10^6\).
Viscous centre modes

Figure 11. Spatial structure of modes B. Solid line: $|p|$; dashed line: Re($p$). (a) Numerical result for the mode $B_0$ indicated in figure 8(a). (b) Theoretical spatial structure (composite approximation in the inner and intermediate regions).

Figure 12. Amplification rates of unstable modes A as function of the Reynolds number, for the set of parameters $m = -2, q = 2, k = 0.5$. Solid lines: numerical results. Dashed lines: theoretical predictions.

6. Conclusion

In this paper, we have performed a large-Reynolds-number asymptotic analysis of the viscous centre modes in an arbitrary vortex with axial flow. By a multiple-scale analysis, general expressions for the frequencies of three families of centre modes have been derived, from which a general instability criterion has been deduced. We have shown that any vortex which satisfies

$$\Omega_0 W_2 \neq 0, \quad (6.1)$$

where $\Omega_0$ is the angular velocity in the centre and $W_2$ is the second radial-derivative of the axial velocity in the centre, is unstable for sufficiently large Reynolds numbers. The axial and azimuthal wavenumbers $k$ and $m$ of the unstable viscous centre modes...
are such that

\[
H_0 = 2\Omega_0 k (2k\Omega_0 - mW_2) < 0. \quad (6.2)
\]

Their growth rates are given at leading order (in dimensional form) by

\[
\sigma^{(0)} \sim \frac{3}{2} \left| \frac{H_0\nu}{4} \right|^{1/3} \quad (6.3)
\]

where \(\nu\) is the kinematic viscosity. The spatial structure of the eigenmodes has also been provided. The theoretical predictions have been compared to numerical results obtained for the \(q\)-vortex model (or Batchelor vortex). Both the frequencies and the eigenmodes have been shown to be well-predicted by the theory.

It is important to emphasize the viscous nature of these modes. They cannot be obtained by an inviscid calculation as near the vortex they exhibit centre radial oscillations on a viscous length scale. In particular, the inviscid centre modes obtained by Stewartson & Brown (1985) and Heaton (2007) cannot be described by the present analysis. Viscous centre modes resemble the Tollmien–Schlichting waves of boundary layers but the physical mechanism explaining the destabilization of some of the centre modes is apparently more complex.

However, these modes can be understood as a phenomenon of viscous wave trapping between two critical points. Critical points, such as turning points of the WKBJ approximations of the viscous solutions (Le Dizès 2004), play the role of boundaries for viscous waves. And, for the centre-mode frequencies, two such critical points are close to each other in the neighbourhood of the vortex centre (in the intermediate region). The frequency selection corresponds to the condition that these two critical points form a waveguide for the viscous waves (see also for instance Morawetz 1952; Chapman 2002). The mathematical structure of the centre modes is indeed typical of double-turning-point problems. In particular, it is very similar to the first ‘bounded states’ of a particle in a parabolic potential well in quantum mechanics, which can also be expressed in terms of Hermite polynomials (see for instance Landau & Lifchitz 1966; Bender & Orszag 1978).

It is worth mentioning that there is \textit{a priori} an infinite number of modes in each family, but that only the first ones satisfying \(n \ll Re^{1/6}\) can be described by the present theory. When \(n\) becomes of order \(Re^{1/6}\) or larger, the two turning points are far apart in the outer viscous region and a different analysis should be carried out in order to describe these modes, as in quantum mechanics for high-energy modes.

In a slightly different asymptotic study, Le Dizès & Fabre (2007) show that vortices with axial flow can also possess viscous ring modes. These modes are discretized by the same ‘double-turning-point’ mechanism but the two turning points in that case are not near the vortex centre but close to a particular radius defined by \(m\Omega_0'(r_c) + kW_0'(r_c) = 0\). For the \(q\)-vortex, these other modes are less unstable than viscous centre modes. However, for other vortex profiles, this is not always the case. In particular, they may be unstable in vortices without axial flow.

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Appendix. General expressions for the eigenmodes

The spatial structure of the eigenmodes is described by expressions (3.14), (3.56) and (3.42) in the outer viscous region, intermediate region and inner region respectively. Relations between the constants \( \hat{A}, \hat{B}, \tilde{A}^{(a)}, \tilde{A}^{(c)}, \tilde{A}^{(2)} \) and \( \tilde{A}^{(3)} \) appearing in these expressions have been derived in § 3.6. With an adequate normalization, they lead, for positive \( K_0 \), to the following expressions (written with the outer variable) for the eigenmodes:

In the inner region,

\[
\hat{p} = \hat{\alpha}_e I_n \left( \mu_0 Re^{1/3} r \right), \quad n \text{ even,} \quad \text{(A 1a)}
\]
\[
\hat{p} = \hat{\alpha}_o \left( (4m^2 + 3)I_n \left( \mu_0 Re^{1/3} r \right) + 8\mu_0 Re^{1/3} r I'_n \left( \mu_0 Re^{1/3} r \right) \right), \quad n \text{ odd,} \quad \text{(A 1b)}
\]

with

\[
\mu_0 = \sqrt{-i\omega_1/3}, \quad \text{(A 2a)}
\]
\[
\hat{\alpha}_e = -\pi 2^{n+1/2} \frac{1}{\Gamma((1 - n/2))} e^{\frac{3i\pi}{2}}, \quad \text{(A 2b)}
\]
\[
\hat{\alpha}_o = -\frac{2^{n-3/2} \beta}{\Gamma(-n/2)} e^{\frac{\pi i}{4} - \frac{i\pi}{2}}. \quad \text{(A 2c)}
\]

In the intermediate region,

\[
\hat{p} = H_n(\beta Re^{1/4} r) \exp \left( -\frac{\beta^2 Re^{1/2} r^2}{4} \right) \frac{(-1)^n \exp(\mu_0 Re^{1/3} r) + i\epsilon(-1)^n \exp(-\mu_0 Re^{1/3} r)}{\sqrt{r}}, \quad \text{(A 3)}
\]

with

\[
\beta = \sqrt{2} e^{\frac{\pi i}{8}} \left( \frac{|K_0|}{6} \right)^{1/4},
\]

and \( \epsilon = 1 \) for modes A and C and \( \epsilon = -1 \) for mode B.

We have used the following relations between the Hermite polynomials \( H_n \) and the parabolic cylinder functions \( D_n(x) \) (Abramowitz & Stegun 1965):

\[
D_n(x) = e^{-x^2/4} H_n(x).
\]

In the outer viscous region,

\[
\hat{p} = \tilde{\alpha} \left( -i\epsilon(-1)^n A^{(2)}[B^{(2)}]^{n+1/2} \exp(Re^{1/6} \eta A^{(2)}) + (-1)^n A^{(3)}[B^{(3)}]^{n+1/2} \exp(Re^{1/6} \eta A^{(3)}) \right), \quad \text{(A 4)}
\]

where

\[
\tilde{\alpha} = -\frac{K_0}{6}^{n/4} 2^{n/2} Re^{(4n+1)/24} e^{\frac{\pi i}{8}}, \quad \eta = 3 \sqrt{\frac{2}{|K_0|}} \frac{H_0}{4}^{1/3},
\]

and the different functions \( A, B \) and \( \Lambda \) only depend on the rescaled variable

\[
s = Re^{1/6} \sqrt{\frac{K_0}{6}} \frac{4}{H_0} r^{1/6}.
\]

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Viscous and inviscid centre modes in the linear stability of vortices: the vicinity of the neutral curves

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In a previous paper, we have demonstrated that if the Reynolds number is sufficiently large, all trailing vortices with non-zero rotation rate and non-constant axial velocity become linearly unstable with respect to a class of viscous centre modes. We provided an asymptotic description of these modes which applies away from the neutral curves in the \((q, k)\)-plane, where \(q\) is the swirl number which compares the azimuthal and axial velocities, and \(k\) is the axial wavenumber. In this paper, we complete the asymptotic description of these modes for general vortex flows by considering the vicinity of the neutral curves. Five different regions of the neutral curves are successively considered. In each region, the stability equations are reduced to a generic form which is solved numerically. The study permits us to predict the location of all branches of the neutral curve (except for a portion of the upper neutral curve where it is shown that near-neutral modes are not centre modes). We also show that four other families of centre modes exist in the vicinity of the neutral curves. Two of them are viscous damped modes and were also previously described. The third family corresponds to stable modes of an inviscid nature which exist outside of the unstable region. The modes of the fourth family are also of an inviscid nature, but their structure is singular owing to the presence of a critical point. These modes are unstable, but much less amplified than unstable viscous centre modes. It is observed that in all the regions of the neutral curve, the five families of centre modes exchange their identity in a very intricate way. For the \(q\) vortex model, the asymptotic results are compared to numerical results, and a good agreement is demonstrated for all the regions of the neutral curve. Finally, the case of ‘pure vortices’ without axial flow is also considered in a similar way. In this case, centre modes exist only in the long-wave limit, and are always stable. A comparison with numerical results is performed for the Lamb–Oseen vortex.

1. Introduction

The stability of trailing vortices with respect to normal mode disturbances is a matter of interest for various applications, such as, for example, the dynamics and dispersal of aircraft trailing wakes. The mapping of the various kinds of waves and instabilities existing in such flows have occupied workers for several decades. In this paper, we focus on particular eigenmodes called centre modes because they are strongly concentrated in the vicinity of the vortex axis.
The existence of centre modes of either a viscous or an inviscid nature has been reported in several numerical and asymptotic studies. Two base flows have received particular attention. First, a systematic study of these centre modes in the case of the swirling Poiseuille flow was conducted by Stewartson, Ng & Brown (1988, hereinafter referred to as SNB). They investigated two regions corresponding to the vicinity of the neutral curves in a Reynolds number/swirl number diagram, and demonstrated the existence of unstable centre modes in these regions. They showed that the most amplified modes are of a viscous nature and gave a description of these modes with a multiple scale analysis. They also showed that these modes coexist with modes of a second kind which are inviscid at leading order but are singular owing to the presence of a critical layer for some real value of $r$.

The second base flow which was particularly considered is the $q$-vortex model (often called Batchelor’s vortex, though it is actually a simplification of Batchelor’s (1964) non-parallel trailing-vortex solution). Until recently, it was generally believed that instabilities exist in this flow only for values of the swirl number $q$ smaller than 1.5. Some recent studies have changed this view. First, Fabre & Jacquin (2004, hereinafter referred to as FJ) have shown that unstable viscous centre modes, related to those existing in the swirling Poiseuille flow, are actually present in this model for all values of $q$ if the Reynolds number is sufficiently large. Figure 1 shows the typical shape of the instability region in the plane defined by the axial wavenumber $k$ and the swirl number $q$, for negative values of the azimuthal wavenumber $m$. The instability region is contained within the domain bounded by the hyperbola of equation $k = -m/q$ and the axes of equations $k = 0$ and $q = 0$. As the Reynolds number tends to infinity, the instability domain tends to fill this region completely.

Le Dizès & Fabre (2007, hereinafter referred to as LDF) subsequently provided an asymptotic description of these modes in the large-Reynolds-number limit. This study is conducted in a general framework and extends the results obtained by SNB for the swirling Poiseuille flow. For the $q$-vortex, the analysis shows that the frequencies of
the unstable modes are given by the following expression:

\[
\omega = mq + k + 3e^{i\pi/6}(kq(kq + m))^{1/3}Re^{-1/3} \\
+ (n + 1/2)e^{-i\pi/4}|6(mq + 2k)|^{1/2}Re^{-1/2} \quad (n = 0, 1, 2, \ldots). \tag{1.1}
\]

(This formula is valid for \( k < -mq/2 \); otherwise a slightly different form has to be employed.) The unstable modes following this formula were called ‘viscous modes of kind A’ by LDF. They also showed that, in addition to these unstable modes, there exist two other families of damped viscous centre modes called viscous modes of kinds B and C. The frequencies of these modes are also given by (1.1), for certain complex choices of the cubic root. Modes of kind B exist in the same region as the unstable modes A, as described in figure 1, and modes of kind C exist outside of this region. The prediction of LDF was successfully compared to numerical results for the \( q \)-vortex for all kinds of modes. However, their description cannot be used in the vicinity of the neutral curves. Inspection shows that in this case the terms of order \( Re^{-1/3} \) and \( Re^{-1/2} \) become comparable in magnitude, so the asymptotic expansion is no longer correctly ordered. The study of these degenerated regions constitutes the primary objective of this paper.

Apart from the viscous modes just discussed, the \( q \)-vortex also possesses eigenmodes of an inviscid nature, and some of them occasionally acquire a centre-mode behaviour. First, unstable inviscid modes exist inside the domain depicted in figure 1. These modes form an infinite family indexed by an integer \( n \). The precise region of existence of these modes was obtained by Heaton (2007) who showed that, unlike what was generally believed, this region extends up to \( q = 2.31 \). Heaton (2007) also showed that the higher modes of this family display a centre-mode behaviour when the index \( n \) becomes large, all the other parameters remaining of order one. It can be noted that these modes also become centre modes in another limit, namely for large values \( |m| \) and in the regions \( q \approx -mk/2 \) and \( q \approx \sqrt{2} \) (Stewartson & Leibovich 1987).

A second category of modes with an inviscid nature is formed by the Kelvin waves, which are neutral modes with a real frequency. Such waves were described for the \( q \)-vortex by Le Dizès & Lacaze (2005), and they generally exist outside of the domain of existence of the viscous modes displayed in figure 1. Leibovich, Brown & Patel (1986, referred to as LBP hereinafter), found that some of these waves behave as centre modes on the external sides of the curves of equation \( k = 0 \) and \( k = -m/q \) bounding the domain of viscous modes in figure 1. Therefore, the curves defined by \( k = 0, k = -m/q \) and \( q = 0 \) delimit the regions of existence of several families of centre modes with either a viscous or an inviscid nature, and interactions between these families can be anticipated in their vicinity. One of the objectives of this paper is to clarify the relations between all these families of centre modes.

Finally, both viscous and inviscid centre modes also exist in ‘pure’ vortices without axial flow. This was demonstrated, in particular, in the numerical study of Fabre, Sipp & Jacquin (2006) for the Lamb–Oseen vortex. The description of the centre modes in this case constitutes the third objective of this paper.

The procedure used here differs from that of LDF for several aspects. First, LDF directly consider the limit \( Re \to \infty \) in the primitive equations with all the other parameters remaining of order one. Here we will consider a distinguished limit, where some parameter (such as the wavenumber or the swirl number) approaches zero as the Reynolds number tends to infinity. Secondly, in LDF, the structure of the eigenmodes is formed of four imbricated layers which have to be matched with each other, whereas in our case the structure of the eigenmode is contained within a single
region. Finally, in LDF, the resolution is made almost entirely in an analytic way. Here, on the other hand, we generally end up with a system of equations which has to be solved numerically. The whole procedure is close to what was performed in §§3 and 4 of SNB for the swirling Poiseuille flow, and the results of SNB are recovered as particular cases of our work.

As in LDF, all the analyses will be conducted in the general case in terms of the Taylor coefficients of the base flow at the centre. However, the general form of the unstable domain given in figure 1 for the \( q \)-vortex remains valid in the general case if the swirl number is defined in a convenient way.

Turning back to figure 1, the neutral curve can be divided into five regions, whose successive study constitutes the summary of this paper (after a general presentation of the base flows and stability equations in §2).

Region 1 is the region of the ‘lower neutral curve’, and is considered in §3.
Region 1’ is the region of the critical swirl number \( q_c \), and is considered in §4.
Region 2 is the region of the ‘upper neutral curve’, and is considered in §5.
Region 3 is the region of the critical wavenumber \( k_c \), and is considered in §6.1.
Region 3’ is the region of the ‘left neutral curve’, and is considered in §6.2.

A comparison of these asymptotic predictions with a numerical solution of the full problem in the case of a \( q \)-vortex is postponed to §7. The case of a pure vortex (without axial velocity) is then considered in §8, and comparisons are made with the numerical results of Fabre et al. (2006) for a Lamb–Oseen vortex. Conclusions are given in §9. Finally, the Appendices regroup analytical developments relevant to some particular limits of the cases treated in the main text.

2. Base flow and stability equations

2.1. Base flows

A columnar vortex is characterized by its angular velocity field \( \Omega(r) = V(r)/r \) and its axial velocity field \( W(r) \). The asymptotic analyses conducted in this paper do not depend on the complete base flow, but only on its properties in the vicinity of the vortex axis. Consequently, we will consider a general base flow defined by its Taylor expansion around the axis, with the following form:

\[
\Omega(r) = \Omega_0 + \frac{\Omega_2 r^2}{2} + O(r^4),
\]

\[
W(r) = W_0 + \frac{W_2 r^2}{2} + \frac{W_4 r^4}{24} + O(r^6).
\]

The terms \( \Omega_0, \Omega_2, W_0, W_2, W_4 \) are assumed to be non-dimensionalized in a proper way by the definition of length and velocity scales. Viscous effects are characterized by a Reynolds number \( Re = 1/\nu \), where the viscosity \( \nu \) is non-dimensionalized using the same scales. For example, for trailing vortices, the usual choice is to define the velocity scale as the difference of axial velocity between the vortex centre and the outer flow, and the length scale as the dispersion radius of axial vorticity. However, other choices of scales are possible and may be more convenient in other contexts. So, as in LDF, we prefer not to favour a particular flow and leave the scalings unspecified.

In the present study, we will generally assume that \( \Omega_0 > 0, \Omega_2 \leq 0 \) and \( W_2 \leq 0 \). These assumptions mean that both the axial velocity and the rotation rate (and, therefore, also the axial vorticity) are maximum at the vortex centre. However, most of the analysis can easily be adapted for other signs of the parameters by symmetry considerations.
It is usual to introduce a swirl number, $q$, which compares the scales of the azimuthal and axial velocities. In the general case where the velocities are given by their Taylor expansions, there are several ways to define such a parameter. Here, we will define the swirl number in the following way:

$$q = \frac{-2\Omega_0}{W_2}. \quad (2.3)$$

For this definition, the ‘upper neutral curve’ will always lie in the vicinity of the hyperbola of the equation $k = -m/q$, whatever the precise vortex model.

For comparison with available work, three specific base flows will be particularly considered. The first one is the $q$-vortex model, used in most studies on trailing vortices (see e.g., FJ), defined by

$$\Omega(r) = q \frac{1 - \exp(-r^2)}{r^2}, \quad W(r) = \exp(-r^2). \quad (2.4)$$

In this case, we have $\Omega_0 = q$, $\Omega_2 = -q$, $W_0 = 1$, $W_2 = -2$, $W_4 = 12$, and the swirl number defined by (2.3) is obviously consistent.

The second particular base flow is the swirling Poiseuille flow, considered in particular by SNB, defined (for $r < 1$) by

$$\Omega(r) = 1, \quad W(r) = \epsilon (1 - r^2). \quad (2.5)$$

In this case, we have $\Omega_0 = 1$, $\Omega_2 = 0$, $W_0 = \epsilon$, $W_2 = -2\epsilon$, $W_4 = 0$. The parameter $\epsilon$ is the Rossby number related to the inverse of the swirl number ($q = \epsilon^{-1}$).

Finally, the third particular base flow is the Lamb–Oseen vortex, whose eigenmodes were mapped by Fabre et al. (2006), defined by

$$\Omega(r) = \frac{1 - \exp(-r^2)}{r^2}, \quad W(r) = 0. \quad (2.6)$$

This latter case corresponds to $\Omega_0 = 1$, $\Omega_2 = -1$, $W_0 = W_2 = W_4 = 0$.

### 2.2. Stability equations

In the stability analysis we consider infinitesimal disturbances in the form of eigenmodes, characterized by an axial wavenumber $k$, an azimuthal wavenumber $m$, and a complex frequency $\omega$, i.e.

$$(u', u'_\theta, u'_z, p') = [u(r), v(r), w(r), p(r)] \exp(ikz + im\theta - i\omega t). \quad (2.7)$$

Linearizing the continuity and momentum equations leads to the following set of equations:

$$\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{im}{r} v + ik w = 0, \quad (2.8a)$$

$$i (m\Omega + kW - \omega) u - 2\Omega v + \frac{\partial p}{\partial r} = \frac{1}{Re} \left[ \left( \Delta_{m,k} - \frac{1}{r^2} \right) u - \frac{2im}{r^2} v \right], \quad (2.8b)$$

$$i (m\Omega + kW - \omega) v + \Xi u + \frac{im}{r} p = \frac{1}{Re} \left[ \left( \Delta_{m,k} - \frac{1}{r^2} \right) v + \frac{2im}{r^2} u \right], \quad (2.8c)$$

$$i (m\Omega + kW - \omega) w + \frac{\partial W}{\partial r} u + ik p = \frac{1}{Re} \Delta_{m,k} w. \quad (2.8d)$$
In these equations, $\Xi(r)$ is the base flow axial vorticity, and $\Delta_{m,k}$ is the Laplacian operator, defined, respectively, as

$$\Xi(r) = \frac{1}{r} \frac{\partial}{\partial r} (r^2 \Omega(r)), \quad (2.9)$$

$$\Delta_{m,k} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - k^2 - m^2/r^2. \quad (2.10)$$

In addition to the eigencomponents $u, v, w, p$, our asymptotic derivations will often make use of the axial vorticity component $\xi$ of the eigenmode, defined by

$$\xi = \frac{\partial v}{\partial r} - \frac{imu}{r}. \quad (2.11)$$

The linearized equation for $\xi$ is the following:

$$(im\Omega + kW - \omega)\xi + \frac{\partial \Xi}{\partial r} u = \Xi ikw - \frac{\partial W}{\partial r} ikv + \frac{1}{Re} \Delta_{m,k}\xi. \quad (2.12)$$

This equation is obtained by combining (2.8b) and (2.8c).

3. The long-wave centre modes

3.1. Asymptotic scalings and equations

We first focus on the long-wave range ($k \approx 0$). Inspection shows that centre modes are always present in this limit for $m \neq 0$, provided that $\Omega_2 \neq 0$, i.e. when the angular velocity field $\Omega(r)$ has a maximum or minimum value at the vortex centre. On the other hand, they do not exist in the swirling Poiseuille flow considered in SNB (1988), where $\Omega(r)$ is constant.

In this section, we will consider trailing vortices with a non-constant axial flow, i.e. with $W_2 \neq 0$ (long-wave centre modes also exist in pure vortices with $W_2 = 0$ and will be considered in §9). To simplify the derivations, we will also assume that $m < 0$ and $\Omega_2 < 0$. Thanks to the symmetries of the problem, a change of sign of either $m$ or $\Omega_2$ leads to identical results, with simply a change of sign of both $k$ and the real part of $\omega$.

For the asymptotic derivation, we require both $k$ and $\omega - m\Omega_0$ to be of order $Re^{-1/2}$, and we apply the least degeneracy principle in order to determine the scalings which allow us to keep the maximum number of terms in the equations. The base-flow parameters are also incorporated into the scalings in order to eliminate them as much as possible in the final equations. The relevant scalings are the following:

$$r = Re^{-1/4} |\Omega_2|^{-1/4} \xi, \quad (3.1a)$$

$$k = Re^{-1/2} |\Omega_2|^{3/2} \Omega_0^{-1} (-W_2/2)^{-1} k, \quad (3.1b)$$

$$\omega = m\Omega_0 + kW_0 + Re^{-1/2} |\Omega_2|^{1/2} \omega + O(Re^{-1}), \quad (3.1c)$$

$$u = u + O(Re^{-1/2}), \quad (3.1d)$$

$$v = v + O(Re^{-1/2}), \quad (3.1e)$$

$$w = Re^{1/4} |\Omega_2|^{-3/4} (-W_2/2) w + O(Re^{-1/4}), \quad (3.1f)$$

$$p = Re^{-1/4} \Omega_0 |\Omega_2|^{-1/4} p + O(Re^{-3/4}), \quad (3.1g)$$

$$\xi = Re^{1/4} |\Omega_2|^{1/4} \xi + O(Re^{-1/4}). \quad (3.1h)$$
Let us first introduce these scalings into the $r$ and $\theta$ momentum equations, (2.8b, c). The equations reduce, at leading order, to

\[
-2v + \frac{\partial p}{\partial r} = 0, \tag{3.2}
\]
\[
2u + \frac{im \rho}{r} = 0. \tag{3.3}
\]

It is instructive to note that the only remaining terms are the Coriolis forces which equilibrate the pressure gradients. This means that, at leading order, the $u$ and $v$ components of the velocity field satisfy the geostrophic balance. Combining these two equations leads to

\[
2\xi = \Delta_m p, \tag{3.4}
\]

where the notation $\Delta_m$ corresponds to the two-dimensional Laplacian operator in terms of the scaled variable $r$, i.e.

\[
\Delta_m = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) - \frac{m^2}{r^2}. \tag{3.5}
\]

This equation, which relates the Laplacian of the pressure to the vorticity field, is also characteristic of the geostrophic regime.

Introducing these scalings into the axial velocity and axial vorticity equation leads to a system of coupled differential equations, which are, respectively, second and fourth order:

\[
-i \left( \frac{mr^2}{2} + \omega \right) w = -im \rho + \Delta_m w, \tag{3.6}
\]
\[
-i \left( \frac{mr^2}{2} + \omega \right) \Delta_m \rho = -4im \rho + 4ikw + \Delta_m^2 \rho. \tag{3.7}
\]

Furthermore, these two equations can be combined into a single, sixth-order differential equation with the following form:

\[
\left( \frac{mr^2}{2} + \omega - i\Delta_m \right) \left[ \left( \frac{mr^2}{2} + \omega - i\Delta_m \right) \Delta_m - 4m \right] \rho + 4km \rho = 0. \tag{3.8}
\]

This equation contains the problem in the most compact form. However, it is nonlinear in $\omega$, and therefore not suitable for a resolution using a global eigenvalue method. In practice, we have solved the pair of equations (3.6)–(3.7), which is linear in $\omega$, with a Chebyshev collocation method similar to that used by FJ and Fabre et al. (2006).

At this stage, we should point out that, through the asymptotic scalings, we have been able to remove all references to the base flow characteristics. Therefore, the results derived in this section are universal and can be applied to all vortex models, provided that the coefficients $\Omega_0, \Omega_2$ and $W_2$ are non-zero constants.

In order to check whether this is a well-posed eigenvalue problem, we have to consider the behaviour of the solutions at $r = 0$ and at $r = \infty$. At the origin ($r = 0$), the use of axisymmetric coordinates leads to a regular singularity. Trying a Taylor expansion, it is found that three independent regular solutions can be constructed. The three other independent solutions are singular and have to be removed. Now, at $r \to \infty$, the problem admits an irregular singularity. Using standard methods (Bender & Orszag 1978), six independent solutions can be constructed, with the following
behaviours: \( r^{\alpha_0}, r^{-\alpha_0}, \exp(\beta_0 r^2) \) (twice), and \( \exp(-\beta_0 r^2) \) (twice). The parameters \( \alpha_0 \) and \( \beta_0 \) are defined as follows:

\[
\begin{align*}
\alpha_0^2 &= m^2 + 8,  \\
\beta_0^2 &= -im/8.
\end{align*}
\]

(3.9)

The two first solutions are algebraic and correspond to ‘inviscid’ solutions because they could also be derived from the corresponding inviscid problem, obtained by neglecting the Laplacian in the first two brackets. The four other solutions are ‘viscous’ solutions with exponential behaviour. Among these six solutions, three of them are growing functions and have to be discarded. Therefore, the behaviour at \( r \to \infty \) leads to three boundary conditions. So, the total number of boundary conditions is 6 and is equal to the order of the differential equation, ensuring that the problem is well-posed and admits a countable infinity of eigenmode solutions.

3.2. The limiting forms for \( |k| \gg 1 \)

Before presenting the numerical solutions of the problem (3.8), we first consider the possible limits for \( k \to \infty \). In terms of the primitive parameters, this limit corresponds to the case \( Re^{-1/2} \ll k \ll 1 \). Therefore, results obtained in this case are expected to match with the predictions obtained by first letting \( Re \to \infty \) with fixed \( k \), and then letting \( k \to 0 \). Two kinds of mode can be predicted in this way.

3.2.1. The viscous limit

The first kind of mode that can be expected for \( k \gg 1 \) is the viscous centre mode described by LDF. When translated into the current scalings, the frequencies of these modes are given by the following formula:

\[
\omega \approx 3|m|k^{1/3}e^{i\gamma} + (n + 1/2)\sqrt{6|m|}e^{-i\pi/4} \quad (n = 0, 1, 2, \ldots).
\]

(3.10)

As shown by LDF, three families of viscous modes, A, B and C, are predicted by this formula, with different values of the angle \( \gamma \). Family A exists for \( k > 0 \), and corresponds to \( \gamma = \pi/6 \). This is the only family which contains unstable modes; the modes of families B and C are all stable. Family B exists for \( k > 0 \) and corresponds to \( \gamma = -\pi/2 \), and family C exists for \( k < 0 \) and corresponds \( \gamma = -\pi/6 \).

3.2.2. The inviscid limit

Inviscid centre modes are also present in the long-wave limit, and can be expected to provide a limit for \( k \gg 1 \) for the solutions of the problem (3.8). Such inviscid centre modes were initially described in the inviscid case by LBP, who referred to them as ‘fast waves’ because their phase velocity \( \omega/k \) tends to infinity as \( k \) tends to zero. These modes are treated in Appendix B, where we recover the results of LBP and extend them to include the leading-order effect of viscosity. In terms of the primitive parameters, the frequencies of these modes are given by the following expression:

\[
\omega = (m\Omega_0 + kW_0) + \frac{W_2\Omega_0}{2\Omega_2}Ck + \frac{i}{Re}\frac{2\Omega_2^2}{W_2\Omega_0}Dk,
\]

(3.11)

where the terms \( C \) and \( D \) take discrete positive values indexed by an integer \( n = 0, 1, 2, \ldots \). The expressions for \( C(m, n) \) and \( D(m, n) \) are given in Appendix B, and some numerical values are displayed in table 5. Note that when expressed in the current scalings, (3.11) takes a much simpler expression:

\[
\omega = Ck + \frac{iD}{k} + O(k^{-3}).
\]

(3.12)
If $k < 0$, the leading-order structure of the eigenmodes, noted $p^{(0)}$ and given by equations (B 6) and (B 12), is regular for all real values of $r$. These modes can be identified as waves propagating along the vortex. The term $C$ can be interpreted as a dimensionless group velocity relative to the frame of the vortex centre. The term $D$ gives the first viscous correction to the frequency of these waves, and leads to a weak damping. The fact that the viscous correction varies as $k^{-1}$ can be explained by noting that as $k$ decreases, the modes becomes increasingly concentrated at the vortex centre. Therefore, the radial gradients are increased, and the effect of viscosity is enhanced. The same observations were made for the long-wave centre modes of the Lamb–Oseen vortex by Fabre et al. (2006) (see also §8 where this case will be reviewed).

The case $k > 0$ leads to completely different conclusions regarding the structure of the modes. In effect, in that case the leading-order structure of the eigenmodes $p^{(0)}$, obtained from the inviscid equations, is singular for some real value of $r$, noted $r_c$ and called the critical point. Eigenmodes with such a structure will be referred to as inviscid singular modes in the following. They do not constitute acceptable solutions of the strictly inviscid problem, and were discarded by LBP who concluded that there are no inviscid centre modes for $k > 0$. On the other hand, they still provide a possible limit for modes computed using the viscous equations, and it will be shown in the next section that they are effectively reached in this way. Note that (3.11) predicts that in the presence of viscosity, these modes become slightly unstable (although they are always much less amplified than the viscous modes considered previously).

Additional details on the structure of the inviscid modes are given in Appendix B. The relation between these modes and other kinds of inviscid centre modes, in particular those investigated in Heaton (2007), remains to be clarified. We may also wonder if, when moving far away from the $k = 0$ limit, these modes remain singular or if they become regular. These issues will be considered in §7.4 for the case of the $q$-vortex.

### 3.3. Numerical results

We now present the numerical solutions $\omega(k, m)$ of the eigenvalue problem (3.6), (3.7), and compare the results to the various limiting cases considered above. We restrict ourselves to the case $m = -1$. The results for $m = -2$ and $m = -3$ show similar trends and are available as a supplement to the online version of the paper.

Numerical results for $m = -1$ are shown in figure 2 with thick lines. The dashed lines correspond to the ‘inviscid’ predictions derived in §3.2 (equation (3.12) with the values of $C$ and $D$ taken in table 1). The dotted lines correspond to the results for the three families of viscous modes A, B and C predicted by (3.10). As can be observed, all of the predicted limits for $|k| \to \infty$ are being attained by some numerical branch. The accordance with the asymptotic limits is good for all kinds of branches, although it requires larger values of $|k|$ for the inviscid branches than for the viscous branches.

The figure shows that the various kinds of branches exchange their identity in the long-wave region in a very intricate way. This exchange occurs in two places. First, for $k \approx 0$, the branches separate into two subsets, the first leading to unstable modes of both viscous modes of kind A and inviscid singular modes, and the second leading to stable viscous modes of kind B. Secondly, for positive and substantially large values of $k$, ‘crossing events’ occur when one unstable branch, initially following the ‘viscous A’ behaviour, crosses the next branch and acquires the ‘inviscid singular’ behaviour. In figure 2, such an event takes place for $k \approx 40$, giving rise to the first branch of ‘inviscid singular’ modes. Similar events occur outside of the range of the figure and give rise
Figure 2. Asymptotic results for long-wave centre modes for $m = -1$. (a) Imaginary part and (b) real part of the frequencies $\omega$. Numerical results are shown by solid lines. Inviscid limits for $k \gg 1$ are displayed with dashed lines and are labelled IR and IS (for regular and singular behaviours) Viscous limits for $k \gg 1$ are shown by dotted lines and are labelled A, B, C. The symbols represent two-dimensional results for 'w' modes (upward-pointing triangles) and 'p' modes (downward-pointing triangles).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$k_c$</th>
<th>$\omega_c$</th>
<th>$A_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>4.730</td>
<td>4.514</td>
<td>0.1408</td>
</tr>
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<td>8.088</td>
<td>0.1142</td>
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<td>$-3$</td>
<td>29.862</td>
<td>12.1726</td>
<td>0.0858</td>
</tr>
</tbody>
</table>

Table 1. Values of the lower neutral point $k_c$, the corresponding frequency $\omega_c$ and the term $A_c$ defined in (4.6) (for $m = -1, -2, -3$).

to the higher branches of this family. This mode-crossing phenomenon has already been observed in the numerical study of the $q$-vortex by FJ, and in the asymptotic study of the swirling Poiseuille flow by SNB. As we will see in the following sections of the present paper, this feature occurs in the vicinity of all the neutral curves.

The results in the two-dimensional case ($k = 0$) are also shown in figure 2 by symbols. This case is treated in Appendix A, where it is shown that there are two distinct series of modes, with frequencies given, respectively, by (A 5) and (A 6). The first ones (downward-pointing triangles in figure 2) are called w-modes because they only have an axial velocity component. The second ones (upward-pointing triangles
Viscous and inviscid centre modes in vortices

in figure 2) are called p-modes because only the pressure component is significant. Note that for \(|m| = 1\), the w- and p-modes correspond to the same frequencies (except the first w-mode which is alone). This degeneracy does not occur for other values of \(m\).

For the mapping of the neutral curves, the most important result is the value \(k_c\) corresponding to the destabilization of the first branch. These values, and the corresponding values of the frequency \(\omega_c\), are given in table 1 for \(m = -1, -2\) and \(-3\). It is important to point out that the values of \(k_c\) are universal and do not depend upon the precise base flow. Therefore, for all kinds of vortices, the lower neutral curve will always be given by

\[
k_{\text{lower}} = Re^{-1/2} \Omega_2^{3/2} \Omega_0^{-1} (-W_2/2)^{-1} k_c.
\]

(3.13)

4. The region of the critical swirl number

In the former section, we have investigated the long-wave regime in the case where the Reynolds number tends to infinity, while the base-flow parameters \(\Omega_0, \Omega_2\), etc. . . all remain of order one. In this section, we consider the region of the critical swirl number (region 1 in figure 1). In this region, the axial wavenumber is both close to the 'upper neutral curve' defined by \(k_{\text{upper}} \approx -m/q\) and close to the lower neutral curve defined by \(k_{\text{lower}} \approx Re^{-1/2} \Omega_2^{3/2} \Omega_0^{-1} |W_2/2|^{-1}\). We therefore expect the following parameter

\[
A = \left( \frac{k_{\text{upper}}}{k_{\text{lower}}} \right)^{2/3} = |\Omega_2| Re^{-1/3} \left( \frac{W_2}{2} \right)^{4/3},
\]

(4.1)

to be of order 1 in this region.

There are several ways to obtain this condition: for instance, \(\Omega_2\) can be large, of order \(Re^{1/3}\) with \(W_2 = O(1)\), or \(W_2\) can be \(O(Re^{-1/4})\) with \(\Omega_2 = O(1)\). For the \(q\)-vortex, \(A = q Re^{-1/3}\), which means that the critical swirl number is \(O(Re^{1/3})\) in this case.

The analysis of region 1' is similar to the lower neutral curve. The same scalings as introduced in (3.1) can be used. However, to distinguish from the development of the former section, we change the notation and write all scaled quantities with a circumplex instead of an underline, i.e. we write \(\hat{r}\) instead of \(r\), etc. . . . Note, however, that the formal dependency with respect to the Reynolds number will be different from (3.1). It will also depend on whether \(W_2\) is small or \(\Omega_2\) is large.

Repeating the analysis of § 3.1, we see that at leading order, the radial and azimuthal momentum equations still lead to the geostrophic balance (3.3), and that the vorticity equation still leads to (3.7). On the other hand, the axial momentum equation now contains an additional term, and takes the following form:

\[
-\left( \frac{1}{2} m \hat{r}^2 + \hat{\omega} \right) \hat{w} = -i (m + A^{3/2} \hat{k}) \hat{p} + \Delta_m \hat{w}.
\]

(4.2)

Combining this equation with (3.7) leads to the following single sixth-order differential equation:

\[
\left( \frac{1}{2} i m \hat{r}^2 + i \hat{\omega} + \Delta_m \right) \left[ \left( \frac{1}{2} i m \hat{r}^2 + i \hat{\omega} + \Delta_m \right) \Delta_m - 4i m \right] \hat{p} = 4 \hat{k} (m + A^{3/2} \hat{k}) \hat{p}.
\]

(4.3)

It turns out that this equation is identical to (3.8), if we simply do the following substitution:

\[
\hat{k} \equiv \hat{k} \left( 1 + \frac{A^{3/2} \hat{k}}{m} \right).
\]

(4.4)
Therefore, the results in the region of the critical swirl number can be directly deduced from those of the long-wave region studied in §3. In particular, a necessary and sufficient condition for instability is that, for some value of $\hat{k}$, we have

$$\hat{k} \left(1 + \frac{A^{3/2} \hat{k}}{m}\right) > k_c. \quad (4.5)$$

This latter condition is met if $A$ verifies the following property:

$$A < A_c \quad \text{with} \quad A_c = |m|^{2/3} 2^{-4/3} k_c^{-2/3}. \quad (4.6)$$

In terms of the base flow parameters, the condition of instability thus reads:

$$|\Omega_2||W_2/2|^{-4/3} < A_c Re^{1/3}. \quad (4.7)$$

The numerical values of $A_c$ are displayed in table 1 for $m = -1, -2, -3$. Again, this condition is universal and provides a necessary and sufficient condition of instability with respect to centre modes, whatever the precise vortex base flow.

To apply these results to the case of the $q$-vortex, we first note that in this case we must use the following scalings:

$$q = A Re^{1/3}, \quad k = Re^{-1/3} A^{1/2} \hat{k}, \quad \omega = mq + k + Re^{-1/3} A^{1/2} \hat{\omega}. \quad (4.8)$$

Thus, the condition of instability is $q < q_c$, with $q_c = A_c Re^{1/3}$. As expected from the numerical simulations (FJ), it is the helical perturbations ($m = -1$) which have the largest critical swirl number:

$$q_c = 0.1408 Re^{1/3}. \quad (4.9)$$

The corresponding wavenumber and frequency are:

$$k_c = 3.550 Re^{-1/3}, \quad \omega_c = -0.1408 Re^{1/3}. \quad (4.10)$$

Note that these results for the $q$-vortex were reported in de Souza (1998).

5. The region of the upper neutral curve ($k \approx -m/q$)

5.1. Asymptotic scalings

We now consider the region of the 'upper neutral curve', i.e. the vicinity of $k \approx -m/q$ (with $q$ given in the general case by (2.3)). As in §3, we look for the asymptotic scalings which permit us to reduce the equations to a convenient form. We assume that all base-flow parameters $\Omega_0, \Omega_2, W_0, W_2, W_4$ (and therefore also the swirl number $q$) are $O(1)$, and suppose that $W_2$ is negative and $\Omega_0$ is positive. The scalings are as follows:

$$r = Re^{-1/4} |W_2/2|^{-1/2} \Omega_0^{-1/4} \bar{r}, \quad (5.1a)$$

$$k = -m/q + Re^{-1/2} |W_2/2|^2 \Omega_0^{-5/2} \hat{k}, \quad (5.1b)$$

$$\omega = m \Omega_0 + k W_0 + Re^{-1/2} |W_2/2| \Omega_0^{-1/2} \hat{\omega} + O(Re^{-1}), \quad (5.1c)$$

$$u = \bar{u}_0 + Re^{-1/2} |W_2/2| \Omega_0^{-3/2} \bar{u}_1 + O(Re^{-1}), \quad (5.1d)$$

$$v = \bar{v}_0 + Re^{-1/2} |W_2/2| \Omega_0^{-3/2} \bar{v}_1 + O(Re^{-1}), \quad (5.1e)$$

$$w = Re^{-1/4} |W_2/2|^{1/2} \Omega_0^{-3/4} \bar{w} + O(Re^{-3/4}), \quad (5.1f)$$

$$p = Re^{-1/4} |W_2/2|^{-1/2} \Omega_0^{5/4} \bar{p} + Re^{-3/4} |W_2/2|^{1/2} \Omega_0^{-1/4} \bar{p}_1 + O(Re^{-5/4}), \quad (5.1g)$$

$$\xi = Re^{1/4} |W_2/2|^{1/2} \Omega_0^{-1/4} \bar{\xi} + O(Re^{-1/4}). \quad (5.1h)$$
Now we inject these scalings into the primitive equations. The radial and azimuthal
equations lead, again, to the geostrophic equilibrium:

\[-2\bar{v}_0 + \frac{\partial \bar{p}}{\partial \bar{r}} = 0,\]
\[2\bar{u}_0 + \frac{i m \bar{p}}{\bar{r}} = 0.\]

(5.2)

The axial momentum equation also leads to (5.3) at leading order. As in §3, these
equations also imply the following relation between the axial vorticity and the
pressure:

\[2\tilde{\xi} = \Delta_m \tilde{p}.\]

(5.4)

For the next steps of the derivation, we require the equations corresponding to the
next order of the radial, azimuthal and axial momentum equations, and also the one
obtained from the leading order of the axial vorticity equation. Those equations are
as follows:

\[i \bar{\Sigma} \bar{u}_0 + Q \bar{r}^2 \bar{v}_0 - 2\bar{v}_1 + \partial_r \bar{p}_1 = \left( \Delta_m - \frac{1}{\bar{r}^2} \right) \bar{u}_0 - \frac{2i m}{\bar{r}^2} \bar{v}_0,\]

\[i \bar{\Sigma} \bar{v}_0 + i L \bar{r}^3 \bar{u}_0 + \frac{2 \bar{u}_1 + im \bar{p}_1}{\bar{r}} = \left( \Delta_m - \frac{1}{\bar{r}^2} \right) \bar{v}_0 + \frac{2i m}{\bar{r}^2} \bar{u}_0,\]

\[i \bar{\Sigma} \bar{w} + 2(Q - L)\bar{r}^3 \bar{u}_0 + i \bar{k} \bar{p} - 2\bar{r} \bar{u}_1 - im \bar{p}_1 = \Delta_m \bar{w},\]

\[i \bar{\Sigma} \bar{\xi} + 2i(\bar{w} + \bar{r} \bar{v}_0) - 4Q \bar{r} \bar{u}_0 = \Delta_m \bar{\xi},\]

\[\bar{\Sigma} = m(1 - Q/2)\bar{r}^2 - \bar{\omega},\]

\[Q = \frac{-4\Omega_0 \Omega_2}{W^2},\]

\[L = Q + \frac{2W_4 \Omega_0^2}{3W_2^2}.\]

(5.6a)

(5.6b)

(5.6c)

The combination \[\bar{r} \times (5.5b) + (5.5c)\] allows us to eliminate \(\bar{p}_1\) and \(\bar{u}_1\), and leads
to the introduction of a new variable defined as \(\bar{w}^* = \bar{w} + \bar{r} \bar{v}_0\). We see, moreover, that
this group of terms also appears in the axial vorticity equation (5.5d). After a few
rearrangements, these two equations lead to the following set of coupled equations:

\[i \bar{\Sigma} \bar{w}^* + i L \bar{r}^2 \bar{p} + i \bar{k} \bar{p} = \Delta_m \bar{w}^* - \Delta_m \bar{p},\]

\[i \bar{\Sigma} \Delta_m \bar{p} + 4imQ \bar{p} + 4im \bar{w}^* = \Delta_m^2 \bar{p}.\]

(5.7a)

(5.7b)

These two equations are linear in the frequency \(\bar{\omega}\) and have been used for the
resolution of the problem in the rest of this section. They can also be combined into
a sixth-order differential equation:

\[(i \bar{\Sigma} - \Delta_m)[(i \bar{\Sigma} - \Delta_m) \Delta_m - 4imQ] \bar{p} + 4mL \bar{r}^2 \bar{p} + 4m \bar{k} \bar{p} - 4im \Delta_m \bar{p} = 0.\]

(5.8)

As in §3, we have been able to reduce the problem to a single differential equation
for the pressure component. However, we can see that the scaling process has been
less successful than in the long-wave region, because the equation still contains a
reference to the base flow through the two parameters \(Q\) and \(L\). Consequently, this
equation will have to be solved on a case-by-case basis. Note that for a \(q\)-vortex, the
base-flow parameters are \(Q = q^2\), \(L = 0\).
There are two interesting limits. The first one corresponds to the limit \( Q \gg 1 \). In this case, (5.8) becomes identical to (3.8) obtained in the long-wave region, if we make the substitution \( r \equiv Q^{-1/4} \), \( \omega \equiv Q^{-1/2} \), \( k \equiv -Q^{3/2} \). So, in this case, we can apply the results of §3 directly. The second interesting limit is obtained for \( Q = L = 0 \). This case applies to the swirling Poiseuille flow, and was considered by SNB in their §3. In their study, the scaled frequency is called \( \Lambda \) and the parameter corresponding to our \( k \) is termed \( \mu \). It turns out that our equations are equivalent to those of SNB (although very different in form) if we make the transformation \( \Lambda = |m|^{-1/2} \omega \), and \( \mu = -|m|^{-5/2} k \).

The case \( Q = L = 0 \) is also expected to provide the correct description of the ‘upper neutral curves’ in the limit of small swirl numbers for a general vortex when \( \Omega_2 \) is not zero (and has the same order of magnitude as \( \Omega_0 \)). However, the condition that the expressions (5.1) remain well-ordered may also impose additional constraints. For instance, for the \( q \)-vortex, the present scaling with \( Q = L = 0 \) will apply only for \( Re^{-1/3} \ll q \ll 1 \). The cases of very small swirl numbers where this condition is no longer verified will be considered in §4.

As for the long-wave range, we first check the consistency of this eigenvalue problem by considering the limit conditions at \( \bar{r} \to \infty \). As in §3, we have four ‘viscous’ solutions with behaviour \( \exp(\pm \beta_1 \bar{r}^2) \) (two solutions for each sign), and two ‘inviscid’ solutions with behaviour \( \bar{r}^{\pm \alpha_1} \). The parameters \( \alpha_1 \) and \( \beta_1 \) are defined as follows:

\[
\alpha_1^2 = m^2 + \frac{8Q}{Q-2} + \frac{16L}{m(Q-2)^2}, \quad \beta_1^2 = -\frac{\text{im}(Q-2)}{8}. \tag{5.9}
\]

Among the four viscous solutions (except for the very singular case where \( Q = 2 \)), two are exponentially growing functions and have to be removed, and the others two are exponentially decaying. The leading-order behaviour of the solution at \( \bar{r} \approx \infty \) is therefore given by the inviscid solutions, i.e.

\[
\bar{p} \approx c_1 \bar{r}^{\alpha_1} + c_2 \bar{r}^{-\alpha_1}. \tag{5.10}
\]

If \( \alpha_1^2 \gg 0 \), the condition \( c_1 = 0 \) leads to a decaying solution, thus allowing for the existence of discrete eigenmodes. (When \( \alpha_1 = 0 \), (5.10) has to be replaced by \( \bar{p} \approx c_1 \log(\bar{r}) + c_2 \), so the condition \( c_1 = 0 \) still selects bounded solutions.) On the other hand, it may happen that \( \alpha_1^2 < 0 \). In such a case, the two roots \( \pm \alpha_1 \) are both imaginary numbers, so the two inviscid solutions are both oscillating functions at \( \bar{r} \approx \infty \), and no condition on \( c_1 \) or \( c_2 \) can lead to a decaying solution. In this case, it is impossible to construct modes located entirely in the vicinity of the vortex centre.

Therefore, there will be some portions of the ‘upper neutral curve’ which cannot be described directly by solving (5.8). More specifically, the condition \( \alpha_1^2 < 0 \) with \( \alpha_1^2 \) given by (5.9b) defines a ‘forbidden interval’ \( Q \in ]Q_c, 2[ \). The lower bound of the interval, \( Q_c \), corresponds to the value where \( \alpha_1 \), given by (5.9), vanishes. The upper bound is for \( Q = 2 \) where \( \alpha_1 \) is infinite. Note that for \( L = 0 \), which corresponds to the \( q \)-vortex, the lower bound is given by

\[
Q_c = 2m^2/(m^2 + 8). \tag{5.11}
\]

In the ‘forbidden interval’, modes existing in the vicinity of the ‘upper neutral curves’ are not centre modes, but also extend into the outer region. The solution in this range would require a matching with an outer solution, to provide the proper balance between the coefficients \( c_1, c_2 \) appearing in (5.10). This has not been undertaken here. Instead, we shall present in §7 a numerical solution of the full problem for a \( q \)-vortex for a value of \( q \) falling within the forbidden interval. Note that in the forbidden
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interval, the hyperbola \( k = -m/q \) also corresponds to the leading order of the upper neutral point for the inviscid instabilities, which are much more amplified than the viscous ones.

5.2. Limit cases for \(|\tilde{k}| \gg 1\)

As in §3, we first consider the possible limits for \(|\tilde{k}| \gg 1\).

5.2.1. The viscous limit

Some of the modes computed with the scaling of the ‘upper neutral curves’ are expected to match with the viscous centre modes of LDF. When expressed into the current scalings, the prediction of LDF is as follows:

\[
\bar{\omega} \approx 3|\bar{m}|^{1/3} \exp(i\gamma_a) + (n + 1/2)|6m(Q - 2)|^{1/2} \exp(i\gamma_b) \quad (n = 0, 1, 2, \ldots).
\]

(5.12)

As in §3, this formula predicts three families of modes. Families A and B are located on the side \( \bar{k} < 0 \); family C is located on the side \( \bar{k} > 0 \). The angles \( \gamma_a \) and \( \gamma_b \) take different values depending on the sign of \( Q - 2 \). For \( Q < 2 \), \( \gamma_a \) is \( \pi/6 \) for modes A, \( -\pi/2 \) for modes B and \( -5\pi/6 \) for modes C, while \( \gamma_b \) is \( -\pi/4 \) for all modes. For \( Q < 2 \), \( \gamma_a \) is \( 5\pi/6 \) for modes A, \( -\pi/2 \) for modes B and \( -5\pi/6 \) for modes C, while \( \gamma_b \) is \( -3\pi/4 \) for all modes.

5.2.2. The inviscid limit

Inviscid centre modes are also present in the vicinity of the ‘upper neutral curve’, and can be expected to give a limit for the numerical results. This case is studied in Appendix C. It is shown that inviscid centre modes exist only outside of the ‘forbidden interval’ defined above, and that their frequencies are given by the expansion:

\[
\bar{\omega} = C_1 \bar{k} + \frac{iD_1}{\bar{k}} + O(\bar{k}^{-3}).
\]

(5.13)

Note that when expressed in terms of the primitive variables, we obtain the following expression (which is valid for all signs of \( \Omega_0, \Omega_2 \) and \( W_2 \)):

\[
\omega = m\Omega_0 + kW_0 + \frac{2\Omega_0^2}{W_2} C_1 \left( k + \frac{m}{q} \right) + \left( -\frac{W_2}{2\Omega_0} \right)^3 \frac{iD_1}{Re} \left( k + \frac{m}{q} \right)^{-1}.
\]

(5.14)

The parameters \( C_1 \) and \( D_1 \) depend upon the wavenumber \( m \), an index integer \( n = 0, 1, 2, \ldots \), and also upon the base-flow parameters \( Q \) and \( L \). The expressions for \( C_1(m, n; Q, L) \) and \( D_1(m, n; Q, L) \) are given in Appendix C. The group velocity coefficients \( C_1 \) turn out to be positive below the forbidden interval (i.e. for \( 0 < Q \leq Q_c \) with \( Q_c \) defined by (5.11)), and negative above this interval (i.e. for \( Q > 2 \)). On the other hand, the coefficients \( D_1 \) are always negative. Therefore, the inviscid modes will be stable for \( \bar{k} > 0 \), and unstable for \( \bar{k} < 0 \). Note that these inviscid centre modes have already been investigated, in the inviscid framework, by LBP. The leading-order terms in our expressions agree with their result, except that they overlooked the dependence of the results with respect to the fourth derivative of the axial velocity field, which appears in our parameter \( L \). However, they applied their results only to base flows such as the \( q \)-vortex where this parameter is zero.

As in the case \( k \approx 0 \) investigated in §3.2.2, the inviscid eigenmodes considered here are of a different nature depending on the sign of \( \bar{k} \). It is found that the modes are regular for \( \bar{k} > 0 \) and are of the ‘inviscid singular’ kind for \( \bar{k} < 0 \). It will be shown
in the next section that they effectively provide the leading order of some branches computed from the viscous equations. The link between these modes and those found in other inviscid studies, in particular Stewartson & Brown (1985) and Heaton (2007) will be considered in §7.4.

5.3. Numerical results

We now present numerical solutions of the eigenvalue problem (5.8). We restrict ourselves to the case $L = 0$, so that the results will apply directly to the $q$-vortex with $q^2 = Q$. Because of the forbidden interval, two ranges of $Q$ have to be considered: the low-swirl range with $Q \in [0, Q_c]$ and the high-swirl range with $Q \in [2, \infty]$. We will present results for $m = -1$ with three values of $Q$, two in the lower range and one in the upper range. The case $m = -2$ has also been considered and leads to similar trends; results are available as a supplement to the online version of the paper.

Results for $Q = 0$ are shown in figure 3. As mentioned above, this case applies to the swirling Poiseuille flow and was treated by SNB. The corresponding figure in SNB is their figure 3. The results are identical except that, since we are using a global eigenvalue method while SNB used a shooting method, we are able to catch a larger number of branches. Figure 4 shows the results for $Q = Q_c = 2/9$. This value corresponds to the lower bound of the ‘forbidden interval’, but our numerical method is still able to provide a reliable solution in this limit case. Finally, figure 5 displays results with $Q = 4$. In all these figures, we have also displayed the expected limits for $|\bar{k}| \to \infty$ with thin dotted and dashed lines, as defined in figure 2.

All these figures display the same general form, and are similar to the results for the long-wave range shown in §3. We observe three families of branches on the ‘unstable side’, (which now corresponds to $\bar{k} < 0$) and two families on the stable side. Moreover, a good agreement with the expected limits is obtained for both viscous and inviscid types of behaviour. As in the long-wave region, a first reorganization of the branches takes place in the region $\bar{k} \approx 0$ (however, a close inspection shows subtle differences between all the displayed cases and the long-wave case investigated previously, regarding the way the various kinds of branches reconnect together). A second reorganisation occurs under the form of mode-crossing events between the ‘inviscid singular’ and ‘viscous A’ branches, occurring for some negative values of $\bar{k}$.

Finally, the upper neutral point is defined by the value $\bar{k}_c$ where the first branch becomes unstable. The value of $\bar{k}_c$ has been computed as a function of $Q$ for $m = -1$ and $-2$ and results are shown in figure 6 (still with $L = 0$). Note that because of the presence of the ‘forbidden interval’, these curves consist of two branches, the first one for the low-swirl range $(0 < Q \leq Q_c)$ and the other one for the high swirl range $(Q > 2)$. The particular values of $\bar{k}_c$ for $Q = L = 0$ are termed $\bar{k}_{c,0}$ and their numerical values for $m = -1, -2, -3$ are given in table 2 as well as the corresponding frequencies. The value for $m = -1$ is in accordance with the value $\mu_0 \approx 3.8$ given by SNB.

In terms of the primitive variables, the ‘upper neutral curve’ is thus given, in the general case, by

$$k_{upper} = \frac{mW_2}{2\Omega_0} + Re^{-1/2}W_2/2\Omega_0^{-5/2}\bar{k}_c,$$  \hspace{1cm} (5.15)

where $\bar{k}_c$ is, in general, a function of the base-flow parameter $Q$ and $L$. 
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Table 2. Numerical values for the upper neutral point $k_{c,0}$ for $Q = L = 0$, and the corresponding frequency $\bar{\omega}$ (for $m = -1, -2, -3$).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$k_{c,0}$</th>
<th>$\bar{\omega}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$-3.842$</td>
<td>$4.095$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-18.920$</td>
<td>$9.029$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$-57.867$</td>
<td>$14.924$</td>
</tr>
</tbody>
</table>

Figure 3. Asymptotic results for upper point scalings with $Q = L = 0$, $m = -1$. Details as in figure 2.

6. The regions of small swirl

In the former section, we have investigated the vicinity of the ‘upper neutral curve’ by letting the Reynolds number tend to infinity and assuming that all base-flow parameters remain of order one. Considering the expressions (5.1a)–(5.1h), which were the starting point of this analysis, we may observe that these expressions constitute well-ordered developments if the quantity $Re^{-1/2}|W_{2/2}|\Omega_{0}^{-3/2}$ is small. In this section, we consider the case where this quantity is of order one or larger. For this purpose, it is convenient to introduce a ‘rescaled swirl number’, defined as

$$\tilde{q} = \Omega_{0}Re^{1/3}|W_{2/2}|^{-2/3}. \quad (6.1)$$

The whole analysis of the previous section can thus be re-interpreted as the case $\tilde{q} \gg 1$. The case $\tilde{q} = O(1)$ corresponds to the ‘critical wavenumber region’ (region 3 in figure 1) and will be considered in §6.1. The case $\tilde{q} \ll 1$ corresponds to the ‘left neutral curve region’ (region 3’ in figure 1) and will be considered in §6.2.
As we shall see, in both of these regions, the base-flow parameter $\Omega_2$ does not show up in the analysis; therefore, results obtained with the $q$-vortex are equivalent to those for the swirling Poiseuille flow. This is also true for more general vortex models provided that $\Omega_2$ is assumed to have the same order of magnitude as $\Omega_0$ (which is a reasonable assumption for a given family of base flows).

6.1. The ‘critical wavenumber’ region (region 3)
We first consider the case where the Reynolds number tends to infinity simultaneously as $\Omega_0$ tends to zero, with the parameter $\tilde{q}$ defined above being of order one. The relevant scalings in this region are:

\begin{align}
  r &= \text{Re}^{-1/3}|W_2/2|^{-1/3}\tilde{r}, \\
  k &= \text{Re}^{1/3}|W_2/2|^{1/3}\tilde{k}, \\
  \omega &= m\Omega_0 + kW_0 + |W_2/2|^{2/3}\text{Re}^{-1/3}\tilde{\omega} + O(\text{Re}^{-2/3}), \\
  u &= \tilde{u} + O(\text{Re}^{-1/3}), \\
  v &= \tilde{v} + O(\text{Re}^{-1/3}), \\
  w &= \tilde{w} + O(\text{Re}^{-1/3}), \\
  p &= |W_2/2|^{1/3}\text{Re}^{-2/3}\tilde{p} + O(\text{Re}^{-1}).
\end{align}
Figure 5. Asymptotic results for upper point scalings with \( Q = 4, L = 0, m = -1 \). Details as in figure 2.

Figure 6. Upper point \( \bar{k}_c \) as function of \( Q \) for \( m = -1 \) (full line) and \( m = -2 \) (dashed line) (with \( L = 0 \)). (a) Small \( Q \) interval (\( 0 < Q < 2m^2/(m^2 + 8) \)). (b) Large \( Q \) interval (\( Q > 2 \)).
They permit us to reduce the general eigenvalue problem (2.8) to the following system:

\[
\begin{align*}
-i(\tilde{\omega} + \tilde{k}\tilde{r}^2)\tilde{u} - 2\tilde{q}\tilde{v} + \partial_r \tilde{p} &= \left(\Delta_m - \tilde{k}^2 - \frac{1}{\tilde{r}^2}\right)\tilde{u} - \frac{2im}{\tilde{r}^2} \tilde{v}, \\
-i(\tilde{\omega} + \tilde{k}\tilde{r}^2)\tilde{v} + 2\tilde{q}\tilde{u} + \frac{im\tilde{p}}{\tilde{r}} &= \left(\Delta_m - \tilde{k}^2 - \frac{1}{\tilde{r}^2}\right)\tilde{v} + \frac{2im}{\tilde{r}^2} \tilde{u}, \\
-i(\tilde{\omega} + \tilde{k}\tilde{r}^2)\tilde{w} - 2\tilde{r}\tilde{u} + ik\tilde{p} &= \left(\Delta_m - \tilde{k}^2\right)\tilde{w}, \\
\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} (\tilde{r}\tilde{u}) + \frac{im}{\tilde{r}} \tilde{v} + ik\tilde{w} &= 0.
\end{align*}
\] (6.3)

This system is identical to the full problem in the case where the base flow is a swirling Poiseuille flow with swirl number \(\tilde{q}\) and Reynolds number unity, except that boundary conditions at a finite distance are replaced by vanishing conditions at infinity. Therefore, in this region, all vortex models display similar results and are equivalent to the swirling Poiseuille flow.

The system (6.3) does not seem to be reducible to a single equation for the pressure component as in the other regions. On the other hand, it can easily be reduced to a system of two equations for the \(u\) and \(v\) components, as was done for example by FJ. Then, it can be solved numerically using the same numerical method as in the other regions, allowing us to compute the eigenfrequencies \(\tilde{\omega}\) as a function of the scaled swirl number \(\tilde{q}\) and the reduced wavenumber \(\tilde{k}\).

Figure 7 shows the marginal stability curves in the \((\tilde{q}, \tilde{k})\) plane, computed for \(m = -1\) and \(m = -2\). This curve defines an equation for the upper neutral point which can be termed \(\tilde{k} = \tilde{k}_{\text{upper}}(\tilde{q})\). The maximum value of this function over all \(\tilde{q}\) is termed \(\tilde{k}_{\text{max}}\). The numerical values of \(\tilde{k}_{\text{max}}\), as well as the corresponding values of \(\tilde{q}\), are given in table 3. The largest value of \(\tilde{k}_{\text{max}}\) is found for \(m = -2\).

In the limit \(\tilde{q} \gg 1\), we expect to recover the expression for the ‘upper neutral curve’ derived in the previous section, i.e. when translated into the current scalings, \(\tilde{k} \approx -m/\tilde{q} + \tilde{k}_{\text{c,0}} \tilde{q}^{-5/2}\). This expression is plotted in figure 7 with dotted lines. The
Viscous and inviscid centre modes in vortices

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$m$ & $\hat{k}_{\text{max}}$ & $\hat{q}$ \\
\hline
$-1$ & 0.1404 & 4.20 \\
$-2$ & 0.1634 & 6.46 \\
$-3$ & 0.1583 & 9.73 \\
\hline
\end{tabular}
\caption{Numerical values for the 'critical wavenumber' $\hat{k}_{\text{max}}$ and the corresponding parameter $\hat{q}$.}
\end{table}

asymptotic expression for $\hat{q} \ll 1$, to be derived in §6.2, is also plotted in a similar way. Both these limits are correctly recovered.

6.2. The 'left neutral curve' region

This last region corresponds to the case of very small swirl numbers. This region was analysed in §4 of SNB in the case of the swirling Poiseuille flow. SNB referred to this case as the 'lower neutral curve' in a Reynolds number/swirl number diagram.

Here, as all results are interpreted in terms of the $(q, k)$ diagram, we find it more appropriate to refer to this case as the 'left neutral curve' region (see figure 1).

This region can be investigated by letting both $q$ tend to zero and $Re$ tend to infinity in the starting equations with $k$ remaining of order one, as was initially done by SNB. However, it is just as simple to start from the system of the previous section and let $\hat{k}$ tend to zero with $\hat{q} \approx \hat{k}^{1/4}$. To recover the expressions of SNB, we introduce the following notation:

\begin{align}
\hat{q} & = (-m)\hat{k}^{1/2}\sigma, \quad \hat{\omega} = \hat{k}^{1/2}\lambda, \quad \hat{r} = \hat{k}^{-1/4}s, \\
\hat{u} & = ig_1, \quad \hat{v} = \bar{h}_1, \quad \hat{w} = \bar{k}^{-3/4}\bar{f}_1, \quad \hat{p} = \bar{k}^{1/4}\bar{p}_1.
\end{align}

Note that in terms of the primitive parameters, the parameter $\sigma$ (which is actually a rescaled swirl number) and the scaled frequency $\lambda$ are defined as follows:

\begin{align}
\sigma & = \Omega_0|W_2/2|^{-1/2}k^{-1/2}(-m)^{-1}Re^{1/2}, \\
\omega & = kW_0 + m\Omega_0 + |W_2/2|^{1/2}k^{1/2}Re^{-1/2}\lambda.
\end{align}

Introducing this notation into (6.3) and letting $\hat{k}$ tend to zero leads directly to system (4.5) of SNB, which is reproduced below:

\begin{align}
(\lambda + s^2)\bar{g}_1 + 2m\sigma\bar{h}_1 + \partial_s\bar{p}_1 & = i\left(\Delta_m - \frac{1}{s^2}\right)\bar{g}_1 - \frac{2im}{s^2}\bar{h}_1, \\
(\lambda + s^2)\bar{h}_1 + 2m\sigma\bar{g}_1 - \frac{m\bar{p}_1}{s} & = i\left(\Delta_m - \frac{1}{s^2}\right)\bar{h}_1 + \frac{2im}{s^2}\bar{g}_1, \\
(\lambda + s^2)\bar{f}_1 - 2s\bar{g}_1 & = i\Delta_m\bar{f}_1, \\
\frac{1}{s} \frac{\partial}{\partial s} (s\bar{g}_1) + \frac{m}{s}\bar{h}_1 + \bar{f}_1 & = 0.
\end{align}

This eigenvalue problem was solved by SNB, and the solutions $\lambda$ as function of $\sigma$ are shown in their figures 10 and 11 for $m = -1$ and $-2$. For the sake of completeness, we reproduce the result for $m = -1$, obtained using the present numerical method, in our figure 8. This plot displays the same general trends as observed in the other regions of the neutral curves. On the 'stable' side $\sigma < 0$, we recognize two series of branches which display either the inviscid behaviour or the viscous C behaviour. On
the ‘unstable’ side $\sigma > 0$, we recognize branches with the viscous A and B behaviour as well as one branch with the inviscid singular behaviour.

Finally, the location of the neutral curve is defined by the value $\sigma_c$ where the first branch becomes unstable. The numerical values of $\sigma_c$ for $m = -1, -2, -3$ as well as the corresponding frequencies are given in table 4. The results for $m = -1$ are in accordance with the value $\sigma_c \approx 6.45$ given in SNB.

Coming back to the primitive parameters, we can note that the location of the left neutral curve is given, whatever the precise base flow, by

$$\Omega_0 W_2 / 2 |^{-1/2} k^{-1/2} (-m)^{-1} Re^{1/2} = \sigma_c.$$  \hspace{1cm} (6.8)

Note that when expressed in the variables of the previous section, this condition takes the much simpler form:

$$\tilde{k} = \left( \frac{\tilde{q}}{m \sigma_c} \right)^2.$$  \hspace{1cm} (6.9)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_c$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>6.425</td>
<td>-5.841</td>
</tr>
<tr>
<td>-2</td>
<td>4.202</td>
<td>-7.592</td>
</tr>
<tr>
<td>-3</td>
<td>4.152</td>
<td>-9.520</td>
</tr>
</tbody>
</table>

Table 4. Values of the ‘left’ neutral point $\sigma_c$ and the corresponding frequency $\lambda$ for $m = -1, -2, -3$. 

Figure 8. Asymptotic results in the ‘left neutral curve’ region (case $m = -1$).
This expression is plotted in figure 7, and gives, as expected, the limit for $\tilde{k} \ll 1$ of the neutral curve.

7. Application to the $q$-vortex

In this section, we apply our asymptotic results to the $q$-vortex, and compare with numerical results directly taken from FJ or obtained with the same numerical method.

7.1. The neutral curves

We first gather the results obtained in the previous sections for the location of the neutral curves.

For the lower neutral curve region:

$$k_{\text{lower}} = \frac{1}{2} Re^{-1/2} q^{1/2} k_c,$$

with $k_c$ given in table 1.

For the critical swirl region:

$$k_c = \left[ 1 + \frac{q k}{m} \right]^{1/2} = k_c.$$

For the upper neutral curve region (outside of the forbidden interval):

$$k_{\text{upper}} = -m/q + Re^{-1/2} q^{-5/2} \tilde{k}_c,$$

where $\tilde{k}_c$ depends upon the swirl number $q$ and is given in figure 6 with $Q = q^2$.

For the critical wavenumber region:

$$k = Re^{1/3} \tilde{k}_{\text{upper}}(\tilde{q}),$$

where $\tilde{q} = Re^{1/3} q$, and the function $\tilde{k}_{\text{upper}}(\tilde{q})$ is defined by figure 7.

For the left neutral curve region:

$$q_{\text{left}} = |m| k^{1/2} Re^{-1/2} \sigma_c,$$

where $\sigma_c$ is given in table 4.

Figure 9 shows the neutral curves in the case $Re = 10^4$, $m = -1$. The thick lines correspond to the numerical results. The thin dotted dashed and dash-dotted lines correspond, respectively, to the approximations (7.2), (7.3) and (7.4). The very thin line shows the leading-order approximation of the 'upper neutral curve', i.e. $k = -m/q$. The approximations (7.1) and (7.5) are not shown since they constitute particular cases of (7.2) and (7.4).

As can be observed, the agreement with the asymptotic prediction is good for the lower and upper neutral curves. The agreement is less good in the critical swirl number and critical wavenumber regions. This is not surprising since the asymptotic analyses in these regions require, respectively, $q \gg 1$ and $k \gg 1$, which are far from being verified here. The left neutral curve is in good agreement with the asymptotics for $k \approx 2$. Note that the numerically obtained neutral curve departs from the asymptotic prediction for $k < 1.4$ and extends into the region of negative swirls. This is due to the existence of inviscid instabilities in this region (Mayer & Powell 1992). This part of the neutral curve is not shown in the figure. The location of the ‘forbidden interval’ of the ‘upper neutral curve’ is also indicated. As can be observed, in this interval, the ‘upper neutral curve’ is given by its leading order, $k = -m/q$, with an excellent approximation.
Figure 9. Neutral curves in the \((k, q)\)-plane for the \(q\)-vortex, for \(Re = 10^4\) and \(m = -1\). Full line, numerical result from Fabre & Jacquin (2004). Dashed lines, equation (7.2). Dotted line, equation (7.3). Dash-dotted line, equation (7.4). Thin line, \(k = -m/q\).

Figure 10. As in figure 9 but for \(m = -2\) and \(Re = 10^5\). (a) Critical wavenumber region; (b) critical swirl number region.

A second comparison case, which corresponds to \(m = -2\), \(Re = 10^5\), is shown in figure 10. Only the critical swirl and critical wavenumber regions are depicted. Here, a reasonable agreement with the asymptotic predictions is observed in both regions.

7.2. Temporal branches

We now consider the temporal branches (i.e. \(\omega\) as function of \(k\)) described in §4.1 of FJ and compare with the various asymptotic predictions. We consider only one case, corresponding to \(q = 2\), \(m = -1\), \(Re = 5 \times 10^5\). Figure 11(a) displays the amplification rate \(\omega_i\), and figure 11(b) displays the relative oscillation rate \(\omega_r - (mq + k)\) (i.e. relative to the frame of the vortex centre). The thick grey lines correspond to the numerical
results taken from figure 5(f) of FJ. The thin lines correspond to the asymptotic results with the lower and the upper neutral curves scalings (from figures 2 and 5 of the present paper). Finally, the dashed lines give the asymptotic prediction from LDF for the two first viscous modes of kinds A and B.

The figures nicely illustrate how the different approximations match together. For $k<0.1$ and $k>0.4$, the numerical results closely follow the results with the neutral curve scalings. On the other hand, in the central region of the plots, the predictions of LDF apply, and give a good estimation to the numerical branches. The agreement with the latter estimate is less good than in the neutral curve regions, but it must be remembered that the analysis of LDF requires $Re^{-1/6} \ll 1$ versus $Re^{-1/4} \ll 1$ for the present analyses.

As a final remark, the branch which displays the ‘inviscid singular’ behaviour in both neutral curve regions actually extend throughout the whole range of $k$. This branch will be considered specifically in §7.4.
Figure 12. Results for the $q$-vortex in the vicinity of the ‘upper neutral curve’ for $q = 1, m = -2$ (in the forbidden interval). Full lines, numerical results for $Re = 10^6$. Dashed lines, predictions for viscous modes A and B. Dotted lines, numerical results in the inviscid case.

7.3. The ‘forbidden interval’

Finally, we have mentioned in § 5.1 that centre modes are not expected to be found in the vicinity of the ‘upper neutral curve’ for values of $q$ lying in a ‘forbidden interval’. To illustrate this case we present (figure 12) the numerical result obtained in the vicinity of $k = -m/q$ for the case $m = -2, q = 1$, which falls within this interval. The full lines correspond to numerical results for $Re = 10^6$. The dotted lines correspond to numerically obtained inviscid results. The dashed lines correspond to the asymptotic prediction of LDF for the first modes of types A and C.

As can be observed, the C mode is approached by a numerical branch on the stable side. On the other hand, the A mode does not seem to be approached. Instead, we observe several unstable branches whose frequencies are predicted well by the inviscid results. These branches correspond to the well-known inviscid instabilities existing in this region of parameters (see e.g. Leibovich & Stewartson 1983; Mayer & Powell 1992). The structure of these latter instabilities near the ‘upper neutral curve’ $k \approx -m/q$ was investigated by Stewartson & Capell (1985) who demonstrated that
for large $|m|$ they become ring modes, with therefore a completely different nature than the centre modes considered here.

Figure 13 shows the shapes of three modes computed on the most unstable branch, for $k = 1.95$, $k = 1.99838$ (which corresponds to the point where this branch becomes neutral) and $k = 2.05$. It can be observed that, even though some of them display strong gradients at the vortex centre, none of these modes is strictly of centre mode type; instead they all extend to the whole vortex core region. This confirms our expectations that centre modes do not exist in this region of the ‘upper neutral curve’.

7.4. The inviscid near-neutral modes existing inside the unstable domain

In §§ 3.2.2 and 5.2.2 (and in Appendices B and C) it has been shown that on the unstable side of the lower and upper neutral curves, there exist some modes which are solutions of the inviscid equations but are singular at a critical point located along the real $r$-axis. It was subsequently verified that these modes are effectively attained as a limit of branches computed from the viscous equations, and that viscosity has a destabilizing effect on them. In this section, we address the significance of these modes in the strictly inviscid case and their relations with centre modes evidenced in other inviscid studies.

For the case of the $q$-vortex, the existence of inviscid centre modes in the regions considered here was investigated using asymptotic techniques by Stewartson & Brown (1985). Their analysis is similar to that reproduced in our Appendices B and C, except that instead of requiring the solutions to vanish away from the central region, they imposed a matching condition with an outer solution. Their study focused mainly on the vicinity of the ‘upper neutral curve’. The main result is their equation (4.15), which can be set in the following form:

$$\omega = (mq + k) - q^2 C_1 \kappa + \frac{A}{B} \exp(-i\alpha_1 \pi) C_2 \kappa^{\alpha_1 + 1},$$

(7.6)

where $\kappa = -(k + m/q)$ is the distance from the neutral curve, $C_1$ is identical to that defined in our equation (C 11), $C_2$ is a positive term depending upon $q$, $m$ and the index $n$ of the family, $A/B$ is a constant arising from the matching with the outer solution and $\alpha_1$ is the same as defined in (5.9), and amounts to $\alpha_1 = \sqrt{m^2 + 8q^2/(q^2 - 2)}$ for the $q$-vortex.
It can be observed that the two first terms in this expression are identical to those obtained in our case. Moreover, the additional term may have a positive or negative imaginary part, depending upon the sign of \((A/B)\sin \alpha_1 \pi\). When this quantity is negative, this expression predicts unstable modes. In this case, the critical point \(r_c\) acquires a small positive imaginary part, and the corresponding mode becomes regular for all real values of \(r\). Stewartson & Brown (1985) showed that this situation occurs in several intervals of \(q\), the largest of all being found for \(|m| = 1\) and extending up to \(q = 2.31\). This prediction was confirmed numerically by Heaton (2007).

When the quantity \((A/B)\sin \alpha_1 \pi\) is positive, (7.6) predicts damped modes. In this case, the critical point \(r_c\) acquires a small negative imaginary part, and the structure of the corresponding eigenmode is singular along a branch cut originating from the critical point and intersecting the real axis. This case was excluded by Stewartson & Brown (1985) who concluded that in such a case there are no inviscid centre modes in this region. However, such singular damped modes may still be significant in a viscous context and provide the leading order of modes computed from the viscous equations. Note that modes with such a peculiar structure are not uncommon in vortex flows: they were found, for example, in the two-dimensional problem by Briggs, Daugherty & Levy (1970), and in the three-dimensional stability of the Lamb–Oseen vortex by Fabre et al. (2006).

According to this discussion it is clear that the modes investigated in Stewartson & Brown (1985) and in the present study all belong to the same family, as their frequencies are identical at leading order. We now wonder, of the viscous correction computed in the present work and the inviscid one computed by Stewartson & Brown (1985), which is the most significant. Let us consider, for example, the case \(q = 2, m = -1, k = 0.45\) (i.e. \(\kappa = 0.05\)) and the first mode of the family (with index \(n = 0\)). In that case, (7.6) applies with \(\omega_1 = \sqrt{17}, C_2 = 3.025, A/B \sin \omega_0 \pi \approx -0.06,\) and leads to \(\omega_i \approx 0.18\kappa^{1/2} \approx 4 \times 10^{-8}\), in agreement with the order of magnitude of the numerical result of Heaton (2007). Our own study predicts \(\omega_i = -(D_1/8)Re^{-1}\kappa^{-1}\) with \(D_1 = -113.26\). Comparing both expressions, it is found that the condition for the viscous correction to be larger than the inviscid one is \(Re \geq 7 \times 10^{10}\). Similar arguments can be invoked in the vicinity of the lower neutral curve. This case was investigated briefly by Stewartson & Brown (1985) who concluded that inviscid unstable modes do not exist in this case. However, their results also allow the existence of singular damped modes in this case. Reiterating their analysis leads to the inviscid prediction \(\omega_i \approx -64\pi/729q^{-4}\kappa^4\). In situations of practical interest (i.e. \(q > 1.5, Re < 10^9\)) this expression always remains several orders of magnitudes smaller than the viscous prediction obtained from (3.11), which corresponds in this case to \(\omega_i \approx DqRe^{-1}\kappa^{-1}\) with \(D = 96/7\).

Therefore, it can be concluded that under circumstances of practical interest, the inviscid prediction of Stewartson & Brown (1985) will be several orders of magnitude smaller than the viscous correction computed herein, and that the modes will be unstable owing to viscous effects, whatever the precise details of the singularity structure of the inviscid solution. Moreover, these modes will always remain much less amplified compared to the viscous modes existing in the same region.

8. Long-wave centre modes in pure vortices

In §3, we have described the centre modes existing in the long-wave range assuming that \(W_2 \neq 0\). Here, we consider the case \(W_2 = 0\), which corresponds to a ‘pure’ vortex
without axial flow. It turns out that, although the scalings are slightly different, the results in this case can actually be deduced directly from those of § 3. We briefly present the asymptotic derivation and apply the results to the Lamb–Oseen vortex.

8.1. Asymptotic scalings

In this case, the scalings that allow us to reduce the equations to the most generic form are:

\[
\begin{align*}
 r &= \Re^{-1/4} |\Omega_2|^{-1/4} \tilde{r}, \\
 k &= \Re^{-1/4} |\Omega_2|^{3/4} |\Omega_0|^{-1} \tilde{k}, \\
 \omega &= \omega_0 + \Re^{-1/2} |\Omega_2|^{1/2} \tilde{\omega} + O(\Re^{-1}), \\
 u &= \tilde{u} + O(\Re^{-1/2}), \\
 v &= \tilde{v} + O(\Re^{-1/2}), \\
 w &= \tilde{w} + O(\Re^{-1/4}), \\
 p &= \Re^{-1/4} \Omega_0 |\Omega_2|^{-1/4} \tilde{p} + O(\Re^{-3/4}), \\
 \xi_z &= \Re^{1/4} |\Omega_2|^{1/4} \tilde{\xi} + O(\Re^{-1/4}).
\end{align*}
\]

Note that the scalings for \( \omega \) and \( r \) are actually the same as in § 3.1, but the scaling for \( k \) is different.

Repeating the analysis of § 3.1 with these scalings, we see that at leading order, the radial and azimuthal momentum equations still lead to the geostrophic balance (3.3) and that the vorticity equation still leads to the same equation as (3.7), i.e.:

\[
-\frac{1}{2} m \tilde{r}^2 + \tilde{\omega} \Delta_m \tilde{p} = -4i m \tilde{p} + 4i \tilde{k} \tilde{w} + \Delta_m^2 \tilde{p}.
\]

(8.2)

On the other hand, the axial momentum equation now takes the form

\[
-\frac{1}{2} m \tilde{r}^2 + \tilde{\omega} \tilde{w} = -i \tilde{k} \tilde{p} + \Delta_m \tilde{w}.
\]

(8.3)

These two equations can be combined into the following sixth-order differential equation:

\[
\frac{1}{2} m \tilde{r}^2 + i \tilde{\omega} + \Delta_m \left[ i \left( \frac{1}{2} m \tilde{r}^2 + i \tilde{\omega} + \Delta_m \right) \Delta_m - 4i m \right] \tilde{p} = -4 \tilde{k}^2 \tilde{p}.
\]

(8.4)

It turns out that this equation is identical to (3.8) with the following substitution:

\[
mk \equiv \tilde{k}^2.
\]

(8.5)

Therefore, the results in the present case can be deduced directly from those obtained in § 3. For \( m < 0 \), the results obtained for \( \tilde{k} < 0 \) are relevant, and have to be applied for both signs of \( \tilde{k} \). This means, in particular, that all modes are stable. For \( m > 0 \), results can be deduced from the symmetry \((m, \omega) \leftrightarrow (-m, -\omega^*)\), leading again to stable modes only.

We can also use the results of § 3.2 to predict the limits for large \( \tilde{k} \). The two possible limits are the inviscid regular modes and the viscous modes of type C. For the inviscid modes, we expect

\[
\tilde{\omega} \equiv \frac{C(m, n) \tilde{k}^2 + \frac{m D(m, n)}{\tilde{k}^2}}{m}.
\]

(8.6)
or, equivalently, in terms of the primitive variables,

$$\omega \approx m\Omega_0 + \frac{C(m, n)}{-\Omega_2 m} k^2 + \frac{1}{1} \frac{m\Omega_2^2 D(m, n)}{Re k^2}.$$  \hspace{1cm} (8.7)

Note that, because of the symmetry $D(m, n) = -D(-m, n)$, the last part of these expressions predicts damping for both $m > 0$ and $m < 0$. For the viscous C modes, we have the following behaviour for $m > 0$:

$$\tilde{\omega} \approx 3|m|^{4/3} |\tilde{k}|^{2/3} e^{-i\pi/6} + (n + 1/2) \sqrt{6|m|} e^{-i\pi/4} \quad (n = 0, 1, 2, \ldots), \hspace{1cm} (8.8)$$

while for $m < 0$ the real part of $\tilde{\omega}$ is reversed.

8.2. Application to the Lamb–Oseen vortex

To illustrate these results, we consider the case of the Lamb–Oseen vortex, as defined by (2.6). A comprehensive analysis of the normal modes of this vortex was given in Fabre et al. (2006). They observed that two families of modes, which they called modes C and V, acquire a centre mode behaviour in the long-wavelength limit. These two families correspond, respectively, to the ‘inviscid regular’ and ‘viscous C’ modes of the present nomenclature.

Numerical and asymptotic results are compared for a single case, corresponding to $Re = 10^5$ and $m = 1$. The damping rates $\omega_i$ and the frequencies $\omega_r$ are displayed, respectively, in figures 14(a) and 14(b). The symbols are full numerical results, and the thin lines are asymptotic results deduced from those plotted in the stable side (i.e. $\tilde{k} < 0$) of figure 14. A close agreement between the numerics and the asymptotic predictions is observed up to $k \leq 0.3$.

9. Summary

The primary objective of this paper was to obtain the marginal stability curves for the family of viscous centre modes in vortices evidenced in FJ and LDF. The study was conducted for the general case of a vortex described by the Taylor coefficients $\Omega_0, \Omega_2, W_0, W_2, \ldots$ of the rotation rate and axial velocity at the vortex centre.
For the purpose of the study, we have been lead to consider successively five regions of the neutral curve. In each region, we have reduced the stability to a generic system which was numerically solved. In all regions but one, the results are universal and apply to all possible vortex models. In particular, the results of SNB, initially derived in the case of the swirling Poiseuille flow, are recovered as particular cases of our study. The most complicated region turned out to be the vicinity of the ‘upper neutral curve’. In this region results are not universal but depend upon two base-flow parameters \( Q \) and \( L \). Moreover, we showed that there is an interval of this ‘upper neutral curve’ where marginal modes are not centre modes but have a more complicated structure.

In practice, the most important result was obtained when considering the critical swirl region. The study of this region allowed us to obtain a necessary and sufficient condition for the occurrence of unstable centre-modes:

\[
|\Omega_2||W_2/2|^{-4/3} < 0.1408Re^{1/3}.
\] (9.1)

This condition is more precise than that given by LDF, who were able to show that a vortex is unstable in the large-Reynolds-number limit whenever \( \Omega_0W_2 \neq 0 \).

In addition to the mapping of the neutral curves, we have investigated the relations between the viscous unstable centre modes and other kinds of centre modes. We have found that in the vicinity of each neutral curve, four other families are always present. Two of them are stable and of viscous nature, and were described by LDF. The third family corresponds to inviscid regular modes, which are weakly damped by viscosity, and are found on the ‘stable’ side of the neutral curves. The last family are called ‘inviscid singular’ modes because their structure is, at leading order, a solution of the inviscid equation, except in the vicinity of a critical point located along the real \( r \)-axis. These modes are unstable and exist in the same regions as the unstable viscous centre modes, but are always much less amplified. In the strictly inviscid case, and for the case of the \( q \)-vortex, these modes are of the same nature as those previously investigated by Stewartson & Brown (1985) and Heaton (2007), which may be regular and unstable or singular and damped. However, we have shown that under situations of practical interest, the amplification rate of these modes is determined by viscous effects whatever the exact details of their inviscid structure.

In each of the regions studied, a complex reorganization between the various families of centre modes occurs. The various kind of branches exchange their identities, from viscous to inviscid, from unstable to stable and vice versa. All possible situations seem to occur, depending on the region considered. Such a behaviour was evidenced for the swirling Poiseuille flow by SNB. Our systematic study showed the universality and the complexity of this behaviour.

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Appendix A. The two-dimensional centre modes

In this Appendix, we describe the centre modes existing in the two-dimensional case, obtained by imposing \( k = 0 \) in either of the analyses of §§3 and 8. We consider, for simplicity, the case of a ‘pure vortex’ and set \( \tilde{k} = 0 \) in (8.2) and (8.3). This leads to
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two decoupled equations with the following forms:

\[-i\left(\frac{1}{2}m\tilde{r}^2 + \tilde{\omega}\right)\tilde{w} = \Delta_m \tilde{w}, \tag{A1}\]

\[-i\left(\frac{1}{2}m\tilde{r}^2 + \tilde{\omega}\right)\Delta_m \tilde{p} = -4im\tilde{p} + \Delta_m^2 \tilde{p}. \tag{A2}\]

Therefore, in the two-dimensional case, we have two distinct kinds of eigenmodes, which we call the w-modes and the p-modes, respectively. Note that in a strictly two-dimensional framework where an axial velocity is not allowed, only the p-modes are relevant.

Equation (A1) governing the w-modes is recognized as an equivalent of the parabolic cylinder equation in cylindrical coordinates. The general solution of this equation can be written as follows:

\[\tilde{w} = s^{|m|} \exp(-s^2)M(a; b; 2s^2), \tag{A3}\]

where \(M\) denotes the confluent hypergeometric function, also called Kummer’s function (Abramowitz & Stegun 1965), and where the terms entering the expression are defined as follows:

\[s = \sqrt{-\frac{i m}{8}} \tilde{r}, \quad a = \frac{|m| + 1}{2} - \frac{\tilde{\omega}}{\sqrt{2im}}, \quad b = |m| + 1. \tag{A4}\]

In the expression (A3), the Kummer function is generally an exponentially growing function, and the expression does not lead to an eigenmode solution vanishing at \(\tilde{r} \rightarrow \infty\). However, when \(a\) is zero or a negative integer, say \(a = -n\), the Kummer function reduces to a polynomial, thus leading to a bounded solution. This condition gives the general expression for the frequencies of the w two-dimensional modes:

\[\tilde{\omega} = \sqrt{2im(1 + |m| + 2n)} \quad (N = 0, 1, 2, \ldots). \tag{A5}\]

Equation (A2) governing the p-modes is a fourth-order equation related to the parabolic cylinder equation. We did not succeed in giving a general solution of this equation in analytic form. However, the numerical solution strongly suggests that the frequencies of the p modes are actually given by:

\[\tilde{\omega} = \sqrt{2im(\sqrt{m^2 + 8} + 1 + 2n)}. \tag{A6}\]

We have verified numerically the validity of this expression for \(m = 1, 2, 3\) and \(n\) up to 20 and found excellent agreement (up to at least 5 decimal places using 150 collocation points).

Note that Bajer, Bassom & Gilbert (2001) have derived and studied the time-dependent version of (A2) (with the frequency \(\tilde{\omega}\) replaced by a time derivative). They were able to give a time-dependent solution to this equation in closed form. The large-time behaviour of their solution is exponential, and corresponds to the first frequency predicted by (A6) (with \(n = 0\)). They also gave the expression of the corresponding eigenmode:

\[\tilde{p} = s^{|m|} M\left(\frac{\sqrt{m^2 + 8} + |m|}{2}, |m| + 1, -s^2\right). \tag{A7}\]

Despite repeated efforts, we have not been able to generalize this expression for the higher-order eigenmodes.
It can be observed that a degeneracy occurs for $|m| = 1$: the p-mode number $n$ has the same frequency as the w-mode number $n + 1$. So, the eigenmodes are expected to come in pairs with the same frequency, except for the first w-mode (with $n = 0$) which is alone. This degeneracy does not occur for $|m| > 1$ where the frequencies of the p and w modes are always distinct.

In the case $W_2 \neq 0$ considered in §3, the scaling for the frequency is the same, so the formulae (A 5) and (A 6) can be used to predict $\omega$. However, the equations obtained for $k = 0$ are not completely decoupled because the equation for $w$ also contains a term involving $p$. Therefore, the p-modes also have a $w$ component.

**Appendix B. The inviscid centre modes in the long-wave range**

Here, we describe the centre modes solutions existing in the long-wave range that are inviscid at leading order. As explained in §3.2.2, such modes can be described by either letting the Reynolds number tend to infinity in the primitive equations (with fixed $k$), expanding the solution in powers of $Re^{-1}$, and then letting $k \to 0$, or by letting $k \to \infty$ in the equations obtained with the scalings of §3. We chose the second way which allows us to start directly from the equations of §3. Therefore, we consider the following expansions for the frequency and the eigenmode:

$$\omega = Ck + \frac{iD}{k} + O(k^{-3}), \tag{B 1}$$

$$p = p^{(0)} + \frac{k^{-2}}{2} p^{(1)}. \tag{B 2}$$

Then, we substitute this expansion into (3.8). At leading order, we obtain:

$$\left(\frac{1}{2}mr^2 + Ck\right)^2 \Delta_m p^{(0)} - 4m \left(\frac{1}{2}mr^2 + k\right) p^{(0)} + 4km p^{(0)} = 0. \tag{B 3}$$

It is then convenient to introduce a new variable $s$, defined in terms of $r$ (and in terms of the primitive variable $r$) as follows:

$$s^2 = \frac{mr^2}{2Ck} \left[= \frac{2\Omega_0^2}{-W_2\Omega_0} \frac{mr^2}{2Ck}\right]. \tag{B 4}$$

With this ansatz, (B 3) is the following compact form:

$$(s^2 + 1)^2 \Delta_m p^{(0)} - 8 \left(s^2 + 1 - \frac{1}{C}\right) p^{(0)} = 0, \tag{B 5}$$

where the notation $\Delta_m$ is now the two-dimensional Laplacian operator operating on the variable $s$. This equation admits a solution in terms of the hypergeometric function $F$ (Abramowitz & Stegun, 1965), with the following form:

$$p^{(0)} = s^{|m|}(1 + s^2)^{-\mu} F(a, b; |m| + 1; -s^2); \tag{B 6}$$

$$\mu(\mu + 1) = \frac{2}{C}, \tag{B 7}$$

$$a = -\mu + \frac{|m| - \alpha_0}{2}, \tag{B 8}$$

$$b = -\mu + \frac{|m| + \alpha_0}{2}, \tag{B 9}$$

with, again,

$$\alpha_0 = \sqrt{m^2 + 8}. \tag{B 10}$$
This expression does not, in general, describe an eigenfunction vanishing at $s \to \infty$. The exception is met when $b$ is zero or a negative integer, say $b = -n$. This condition gives the allowed values for the group velocity coefficient $C(m, n)$:

$$C(m, n) = \frac{8}{(|m| + \sqrt{m^2 + 8 + 2n})(|m| + \sqrt{m^2 + 8 + 2n} + 2)}.$$  \hfill (B 11)

Furthermore, in such cases, the hypergeometric function reduces to a polynomial. For example, we detail the structure of the eigenmodes corresponding to the three lowest values of $n$:

$$p^{(0)}(s) = \frac{s^{|m|}}{(1 + s^2)^{|m|/2 + \alpha_0/2}} \quad (n = 0),$$  \hfill (B 12a)

$$p^{(0)}(s) = \frac{s^{|m|}}{(1 + s^2)^{|m|/2 + \alpha_0/2 + 1}} \left(1 - \frac{\alpha_0 + 1}{|m| + 1} s^2 \right) \quad (n = 1),$$  \hfill (B 12b)

$$p^{(0)}(s) = \frac{s^{|m|}}{(1 + s^2)^{|m|/2 + \alpha_0/2 + 2}} \left(1 - \frac{\alpha_0 + 2}{|m| + 1} s^2 + \frac{(\alpha_0 + 1)(\alpha_0 + 2)}{(|m| + 1)(|m| + 2)} \right) s^4 \quad (n = 2).$$  \hfill (B 12c)

From either of these forms, it is clear that the eigenfunctions behave, as $s \to \infty$, as $s^{-\alpha_0}$, which is the expected behaviour for inviscid modes according to the discussion in §3.1.

Up to this point, our results are in complete agreement with those derived by LBP in the inviscid framework. To describe the effect of viscosity, we must consider the next order of the expansion of (3.8) in powers of $k^{-1}$, which gives:

$$(s^2 + 1)^2 \Delta_m p^{(1)} - 8 \left(s^2 + 1 - \frac{1}{C} \right) p^{(1)} = -\frac{D}{C} (2(s^2 + 1) \Delta_m p^{(0)} - 8 p^{(0)})$$

$$+ \frac{1}{2C^2} \left( (s^2 + 1) \Delta^2_m p^{(0)} + \Delta_m (s^2 + 1) \Delta_m p^{(0)} - 8 \Delta_m p^{(0)} \right).$$ \hfill (B 13)

This equation defines a differential equation for $p^{(1)}$ similar to that obtained for $p^{(0)}$ at the lowest order, with a forcing term, defined in terms of $p^{(0)}$, on the right-hand side. To obtain a well-behaved solution, the left-hand side must be orthogonal to the homogeneous solution of the equation. This condition determines the values of the coefficient $D(m, n)$. The derivation is standard, and leads to the general expression:

$$D(m, n) = \frac{m}{2C(m, n)} \int_0^\infty \frac{p^{(0)}(s)}{(1 + s^2)^2} \left( (s^2 + 1) \Delta_m p^{(0)} + \Delta_m (s^2 + 1) \Delta_m p^{(0)} - 8 \Delta_m p^{(0)} \right) s \, ds$$

$$\int_0^\infty \frac{p^{(0)}(s)}{(1 + s^2)^2} \left( 2(s^2 + 1) \Delta_m p^{(0)} - 8 p^{(0)} \right) s \, ds.$$ \hfill (B 14)

The integrals appearing in this expression can be evaluated analytically (Abramowitz & Stegun 1965). Here, we give the expression only for the first mode of the family, with $n = 0$:

$$D(m, 0) = \frac{|m|(|m| + 1)\mu_0^2(2\mu_0 + 1 - |m|)}{2(2\mu_0 + 3)(|m|\mu_0 + \mu_0 + 2)} \left((2 + |m|)\mu_0^2 + (7 + 2|m|)\mu_0 + 6\right),$$ \hfill (B 15)

with $\mu_0 = (\sqrt{m^2 + 8 + |m|})/2$. Similar expressions can be obtained for higher values of $n$, but are much more intricate.

Numerical values of $C(m, n)$ and $D(m, n)$ for $m = -1, -2, -3$ and $n = 0, 1, 2$ are given in table 5. Note that for $m < 0$, the coefficients $C(m, n)$ and $D(m, n)$ are always
positive. Results for \( m > 0 \) can be deduced from the relations \( C(-m, n) = C(m, n) \) and \( D(-m, n) = -D(m, n) \).

Let us consider in more detail the structure of the ‘inviscid solution’ \( p^{(0)} \) given by (B 6) and (B 12). Because of to the factor \((1 + s^2)^{-\mu}\), this solution is singular when \( s^2 = -1 \). When \( k < 0 \), the change of variables defined by (B 4) leads to positive values of \( s^2 \) when \( r \) is real, so singularities do not occur on the real axis. In this case, the modes are regular. On the other hand, if \( k > 0 \), a singular point is found on the real \( r \)-axis, and occurs at the position \( r_c = \sqrt{-2Ck/m} \). This situation leads to the prediction of inviscid singular modes. Such modes were discarded by LBP, who concluded that inviscid centre modes exist only for negative \( k \). However, it is shown in § 3.3 that centre modes of the present type are actually reached as a limit by some branches computed with the scaling of § 3. Note that modes with a similar structure were also found by SNB for the swirling Poiseuille flow.

Technically, there are two ways of justifying the continuation of the eigenmode structure through this critical layer. The first way would be to introduce the viscosity in a region neighbouring the singularity called the critical layer, and to solve an inner problem whose solution should be matched to the outer singular expression for (B 6). This argument was carried out by SNB in a related case. In the present case, it can be expected that the width of the critical layer is of order \( r - r_c = O(k^{-1/3}) \) (or \( r - r_c = O(Re^{-1/3}) \) in primitive variables). Alternatively, we may argue that since \( \omega \) has a small positive imaginary part, the critical point \( r_c \) is actually slightly below (resp. above) the real axis for \( m > 0 \) (resp. \( m < 0 \)). This is consistent with the rule given in Olendraru et al. (1999) which states that the contour of integration must lie above (resp. below) the critical point for \( m > 0 \) (resp. \( m < 0 \)).

### Appendix C. The inviscid centre modes in the region of the upper neutral curve

As for in the long-wave range, we consider the inviscid limit for centre modes in the vicinity of the ‘upper neutral curve’ by letting \( \tilde{k} \to \infty \) in the reduced equation (5.8). We consider the following expansion for the frequency and eigenmode:

\[
\bar{\omega} = C \bar{k} + \frac{iD}{\bar{k}} + O(\bar{k}^{-3}),
\]

\[
\bar{p} = \bar{p}^{(0)} + \bar{k}^{-2} \bar{p}^{(1)}.
\]

It is also convenient to introduce a new variable \( s \), defined in terms of \( \bar{r} \) as follows:

\[
s^2 = \frac{m(2 - Q)\bar{r}^2}{2C \bar{k}}.
\]
Introducing this ansatz into (5.8) and retaining only the leading-order term leads to the following compact equation:

\[
(s^2 + 1)^2 \Delta_m \bar{p}^{(0)} - \frac{8Q}{Q-2} (s^2 + 1) \bar{p}^{(0)} - \frac{16L}{m(Q-2)^2} s^2 \bar{p}^{(0)} - \frac{8}{(Q-2)C_1} \bar{p}^{(0)} = 0. \tag{C 4}
\]

This equation is similar to the one obtained in the long-wave region. It admits a solution in terms of the hypergeometric function \( F \) with the following form:

\[
\bar{p}^{(0)}(s) = s^{|m|}(1 + s^2)^{-\mu} F(a_1, b_1; |m| + 1; -s^2); \tag{C 5}
\]

with

\[
\mu(\mu + 1) = -\frac{2}{(Q-2)C} + Y/4, \tag{C 6}
\]

\[
a_1 = -\mu + \frac{|m| - \alpha_1}{2}, \tag{C 7}
\]

\[
b_1 = -\mu + \frac{|m| + \alpha_1}{2}, \tag{C 8}
\]

\[
Y = \frac{16L}{m(Q-2)^2}, \tag{C 9}
\]

\[
\alpha_1^2 = m^2 + \frac{8Q}{Q-2} + Y. \tag{C 10}
\]

If \( \alpha_1^2 \geq 0 \), imposing \( b_1 = -n \) allows us to obtain a bounded solution. This leads directly to the following expression for \( C_1 \):

\[
C_1(m, n; Q, L) = \frac{8}{2 - Q} (|m| + \alpha_1 + 2n)(|m| + \alpha_1 + 2n + 2) + Y. \tag{C 11}
\]

On the other hand, in the ‘forbidden interval’ defined by \( \alpha_1^2 < 0 \), the asymptotic behaviour of (C 5) has the form \( \bar{p}^{(0)} \approx \sin(|\alpha_1| \log s + \phi) \), and no condition can lead to a bounded solution. This confirms that inviscid centre modes do not exist in this case.

For \( \alpha_1^2 > 0 \), the term \( D_1(m, n) \) can be computed in the same way as was done in the long-wave range. The result is as follows:

\[
D_1 = \frac{m}{2C_1} \int_0^\infty \frac{\bar{p}^{(0)}}{1 + s^2} \left( (s^2 + 1)\Delta_m \bar{p}^{(0)} + \Delta_m (s^2 + 1)^2 \bar{p}^{(0)} - \frac{8Q+1}{Q-2} \Delta_m \bar{p}^{(0)} \right) s \, ds \int_0^\infty \frac{\bar{p}^{(0)}}{1 + s^2} \left( 2(s^2 + 1)\Delta_m \bar{p}^{(0)} - \frac{8Q}{Q-2} \bar{p}^{(0)} \right) s \, ds. \tag{C 12}
\]

Again, the integrals appearing in this expression can be worked out analytically, leading, for the first mode of the family, to

\[
D_1(m, 0) = \frac{-(Q-2)^2|m|(2|m| + 1 - |m|)}{2(2\mu_1 + 3)((Q-2)(|m| + 1)\mu_1 + 2Q)} \\
\times ((2+|m|)(Q-2)\mu_1^2 + 2|m|(Q-2) + 7Q-8)\mu_1 + 6Q). \tag{C 13}
\]

with \( \mu_1 = (\alpha_1 + |m|)/2. \)

Note that, unlike in the previous section, these coefficients are not universal, but depend on the base flow parameters \( Q \) and \( L \). Moreover, the predictions are valid only outside of the ‘forbidden interval’. In the case \( L = 0 \), the term \( C_1 \) is positive.
below the forbidden interval (i.e. for $Q \leq Q_c = 2m^2/(m^2 + 8)$), and negative above it (i.e. for $Q > 2$). The term $D_1$ is always negative. This means that the present analysis leads to stable regular modes on the ‘stable’ side of the neutral curve (for $k > -m/q$), and to unstable singular modes (i.e. with a critical layer) on the ‘unstable’ side of the neutral curve (for $k < -m/q$).

Finally, we give the limiting form of expressions (C11) and (C13) obtained for some particular values of $Q$ (with, again, $L = 0$).

(i) $Q = 0$

This case corresponds to the swirling Poiseuille flow already investigated by SNB. In this case we have:

$$C_1(m, n) = \frac{1}{(|m| + n)(|m| + n + 1)},$$

$$D_1(m, 0) = \frac{2m^2(|m| + 1)(|m| + 2)^2}{2|m| + 3}.$$  

These results correspond, respectively, to expressions (3.24) and (3.27) of SNB (except that a different notation is used).

(ii) $Q = Q_c = 2m^2/(m^2 + 8)$

This value corresponds to the lower bound of the ‘forbidden interval’. It is found that in this case the analysis is still valid, and leads to the following predictions:

$$C_1(m, n) = \frac{m^2}{(|m| + 2n) (|m| + 2n + 2)},$$

$$D_1(m, 0) = \frac{-8|m|^3(|m| + 1)(m^2 + 2|m| + 32)}{(m^2 + 8)^2(|m| + 2)(|m| + 3)}.$$  

(iii) $Q \to 2^+$

This is the upper bound of the ‘forbidden interval’. In this limit, the terms $C_1$ and $D_1$ have the following behaviour:

$$C_1(m, n) \approx -\frac{1}{2} + \frac{|m| + 1 + 2n}{4} \sqrt{Q - 2} + O(Q - 2),$$

$$D_1(m, 0) \approx -6|m|(1 + |m|) \sqrt{Q - 2} + O(Q - 2).$$  

(iv) $Q \to \infty$

$$C_1(-m, n) \approx -Q^{-1}C(m, n),$$

$$D_1(-m, n) \approx -Q^2D(m, n),$$

where $C(m, n)$ and $D(m, n)$ are the expressions for the long-wave centre modes given in the previous section. As expected, in this limit, the results in the region of the upper curve become equivalent to those in the long-wave range.

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On vortex rings around vortices: an optimal mechanism

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Stable columnar vortices subject to hydrodynamic noise (e.g. turbulence) present recurrent behaviours, such as the systematic development of vortex rings at the periphery of the vortex core. This phenomenon lacks a comprehensive explanation, partly because it is not associated with an instability stricto sensu. The aim of the present paper is to identify the physical mechanism triggering this intrinsic feature of vortices using an optimal perturbation analysis as a tool of investigation. We find that the generation of vortex rings is linked to the intense and rapid amplification of specific disturbances in the form of azimuthal velocity streaks that eventually evolve into azimuthal vorticity rolls generated by the rotational part of the local Coriolis force. This evolution thus appears to follow a scenario opposite to the classical lift-up view, where rolls give rise to streaks.

1. Introduction

Vortices are ubiquitous in fluid flows, appearing in the tiniest scales of turbulence as well as in the largest geophysical ones. They have been the focus of considerable research effort over the past few decades. Partly motivated by industrial concerns related to the potential hazard of wing-tip vortices to following planes (e.g. see Stafford 2006), most of these studies aimed to depict the modal instabilities developing in vortices. As a result, several stability criteria have been elaborated in order to identify the candidates for an instability of centrifugal, inflectional (Gallaire & Chomaz 2003) or elliptic nature (Kerswell 2002). But some generic behaviours systematically exhibited by vortices still lack a convincing description. One can cite vortex bursting, a localized ‘turbulent burst’ affecting the vortex core while possibly travelling (Spalart 1998; Moet et al. 2005), vortex meandering, an erratic displacement of the vortex core occurring at large wavelengths (Devenport et al. 1996; Jacquin et al. 2005), or the systematic development of vortex rings at the periphery of vortices in both numerical simulations (Melander & Hussain 1993) and experiments (Beninati & Marshall 2005, and the references therein).

Similarly, in the context of plane shear flows, a phenomenon exhibited as a typical response of the flow in the presence of noise has long failed to be described with standard modal analyses. More specifically the emergence of elongated longitudinal velocity structures, or streaks, in response to an external forcing, is an intrinsic feature of shear flows. Furthermore, these streaks appear to play a key role in the transition process (Schmid & Henningson 2001). While standard analyses have failed in depicting

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the mechanism of emergence of these recurrent patterns, alternative ideas, coined ‘non-modal’ in contrast to classical analyses, have reconsidered the concept of stability. Basically, by allowing disturbances to continuously deform while growing, these approaches have succeeded in identifying ‘optimal’ initial conditions that maximize their energy growth at a given time. Interestingly, these optimal disturbances exploit energy amplification mechanisms that are filtered out with traditional approaches. In particular, one of these mechanisms, the so-called ‘lift-up effect’ (Landahl 1975) is classically invoked to explain the emergence of streaks. This effect appears when studying the dynamics of an optimal disturbance in the form of rolls (of crossflow velocity components) which transform into strong streaks (or streamwise velocity variations). As such amplification mechanisms exploit specific properties of shear flows, one may expect that other mechanisms arise in the context of vortices, and are responsible for non-modal behaviours. Such behaviours were reported in Smith & Montgomery (1995) and Nolan & Farrell (1999) for two-dimensional vortex flows. Recently, Antkowiak & Brancher (2004) carried out an optimal perturbation analysis to examine these phenomena in the three-dimensional case, and found an intense core contamination mechanism. Pradeep & Hussain (2006) pursued this analysis and found evidence of axisymmetric transient growth for a non-diffusive vortex. They interpreted this behaviour in terms of tilting and stretching of vorticity (Kawahara et al. 1997), in line with the classical view of the emergence of vortex rings by the wrapping up of external vorticity and the merger of the wrapped-up structures.

Our goal in this work is to revisit the understanding of vortex rings formation at the periphery of vortices, a phenomenon so frequent that some simulations have mimicked the influence of turbulence on a vortex column by adding such surrounding vortex rings (Marshall 1998). Using tools that have proven to be successful in another context, we identify a simple physical mechanism that optimally promotes the emergence of vortex rings. This mechanism is specific to rotating flows and appears to act in a fashion that is the reverse of the classical lift-up effect, so that streaks (here corresponding to azimuthal velocity variations) now generate rolls (here azimuthal vortex rings) as an outcome. We therefore suggest that vortex rings emerge around vortices just as streaks naturally appear in shear flows.

2. A disturbed vortex

The flow considered in this article is the classical Lamb–Oseen Gaussian vortex, spreading under the action of viscosity. Using the characteristic scales $\Omega_0$, the initial angular velocity at the vortex centre, $r_0$, the initial dispersion radius of the vortex, and $\nu$, the kinematic viscosity, we can express the vorticity $Z(r, t)$ and the angular velocity $\Omega(r, t)$ in the following non-dimensional form:

$$Z(r, t) = \frac{2}{1 + 4t/Re} \exp(-r^2/(1 + 4t/Re)), \quad \Omega(r, t) = \frac{1 - \exp(-r^2/(1 + 4t/Re))}{r^2}.$$  

The Reynolds number defined with these characteristic scales,

$$Re = \frac{\Omega_0 r_0^2}{\nu} = \frac{\Gamma}{2\pi \nu},$$

is the only control parameter of the flow. Here, $\Gamma$ is the circulation of the vortex.

In order to identify the perturbation with the most growth, the evolution equations for an arbitrary disturbance have first to be derived (Fabre, Sipp & Jacquin 2006). In the classical cylindrical coordinates $(r, \theta, z)$, a general hydrodynamic disturbance is
described by its velocity field $u = (u_r, u_\theta, u_z)$ and pressure $p$. Expressing $u_z$ and then $p$ as functions of $u_r$ and $u_\theta$, it is possible to write the whole initial value problem for this reduced set of variables:

$$F(u) = L \frac{\partial u}{\partial t} + Cu - \frac{1}{Re} Du = 0 \quad (2.2)$$

Restricting the scope of this study to axisymmetric disturbances, and considering normal mode disturbances of the form $f(r, t)e^{ikz}$, the following expressions for operators $L$, $C$ and $D$ hold:

$$L = \begin{pmatrix} \delta_k & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -2\Omega Z \\ Z & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \Delta k \delta_k & 0 \\ 0 & \Delta_k \end{pmatrix}, \quad (2.3)$$

where

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} - k^2 \quad \text{and} \quad \delta_k = 1 - \frac{1}{k^2} \Delta_0. \quad (2.4)$$

For a given disturbance, classically we consider the kinetic energy contained in a cylinder of height one wavelength and of infinite radius. Normalized with the factor $k/4\pi^2$, it can be expressed as:

$$E(t) = \frac{1}{2} \int_0^\infty \left( \bar{u}_r u_r + \bar{u}_\theta u_\theta + \bar{u}_z u_z \right) r \, dr = \frac{1}{4} \langle Lu, u \rangle \quad (2.5)$$

where an overbar stands for complex conjugation, and where the scalar product $\langle a, b \rangle$ is defined as $\int_0^\infty \bar{a}^T b \ r \, dr + \text{c.c.}$

3. Some technical background

In the present work, the general framework for the optimal perturbation identification described in Corbett & Bottaro (2001) has been adopted. This optimal control-theory-based strategy presents the advantage of not requiring the steadiness of the flow. As this technique is now classical, only a brief outline will be presented in the following.

A classical variational formulation of the problem is employed to identify the initial condition inducing the most growth. To this end, it proves useful to introduce $u_0$, the form of the perturbation at initial time of energy $E_0$, so that $u$ and $u_0$ are now related through the relation

$$H(u, u_0) = u(0) - u_0 = 0. \quad (3.1)$$

The optimal perturbation is then the particular initial condition maximizing the energy growth

$$G(\tau) = \frac{E(\tau)}{E_0} \quad (3.2)$$

at a given time $\tau$. Equivalently, the optimal perturbation is the particular $u_0$ maximizing the functional

$$\mathcal{J}(u_0, u) = G(\tau), \quad (3.3)$$

subjected to the constraints $F(u) = 0$ and $H(u, u_0) = 0$. The constrained optimization problem is circumvented by introducing the Lagrange functional

$$\mathcal{L}(u, u_0, a, c) = \mathcal{J}(u, u_0) - \langle F(u), a \rangle - \langle H(u, u_0), c \rangle. \quad (3.4)$$

already including the constraints by means of appropriate Lagrange multipliers. Here the scalar product $\langle a, b \rangle$ is defined as $\int_0^\tau \int_0^\infty \bar{a}^T b \ r \, dr \, dt + \text{c.c.}$.
At a stationary point, the directional derivatives of $L$ vanish. More specifically, the cancellation of these derivatives with respect to $a$ and $c$ allows the constraints (2.2) and (3.1) to be recovered, as expected. The condition for the gradient with respect to $u$ to vanish implies that $a$ satisfies the following adjoint equation:

$$F^+(a) = -L \frac{\partial a}{\partial t} + C^+ a - \frac{1}{Re} D a = 0,$$

revealing this Lagrange multiplier as an adjoint variable. Here operator $C^+$ is expressed as

$$C^+ = \begin{pmatrix} 0 & Z \\ -2\Omega & 0 \end{pmatrix}. \quad (3.6)$$

Cancelling the $u$ derivative of $L$ provides transfer relations between the adjoint and direct spaces. Finally, the gradient of the functional with respect to the control can be written

$$\nabla_{u_0} I = -2 \frac{E(\tau)}{E_0^2} u_0 + a(0). \quad (3.7)$$

For a given $u_0$, $a(0)$, computed at the expense of successive integrations of the direct and adjoint equations, allows the functional gradient to be determined. Different strategies can then be elaborated to improve the control, for example the elementary fixed step gradient procedure used in this study.

The preceding approach is designed to identify the initial condition maximizing energy growth at a fixed time $\tau$. A second step may consist in looking for the optimal time $\tau_{opt}$, the result of an optimization with respect to time. The corresponding initial condition is then called a global optimal perturbation, or, in short, the optimal perturbation. In some cases, it will be of interest to find the optimal perturbation corresponding to fixed time $\tau < \tau_{opt}$. In that case, the corresponding optimal perturbation will be termed a short-term optimal (Corbett & Bottaro 2001).

In the numerical treatment of the problem, the spatial derivatives are approximated with a Chebyshev pseudospectral method (e.g. Fornberg 1995), where the Gauss–Lobatto grid is algebraically mapped onto the semi-infinite physical space. The parity of the functions is taken into account in the expression of the derivatives (Kerswell & Davey 1996). All calculations are carried out using MATLAB and the DMSuite package developed by Weideman & Reddy (2000).

4. Axisymmetric amplification

4.1. Quantitative overview

Figure 1 reports the maximal growths achieved with axisymmetric disturbances. It can already be seen that considerable energy amplification (of order of $10^4$) is obtained, though asymptotic exponential decay is predicted with modal analysis. The plot also reveals that large amplification occurs especially at large axial wavelengths. In this limit, the amplification is accompanied by a tendency for the optimal time $\tau_{opt}$ to diverge.

4.2. A note on the $k \to 0$ limit

The behaviour of energy growth near the bidimensional limit $k = 0$ deserves comment. It is useful to stress again that unlike classical linear stability analyses, the present study does not require ‘freezing’ the diffusion of the base flow. With viscous diffusion taken into account, the present results are therefore valid whatever the time scale of amplification (which can be of order of a thousand periods of rotation of the vortex
Figure 1. (a) Maximal amplification reached with axisymmetric optimal perturbations, as a function of the axial wavenumber $k$. (b) Corresponding optimal time.

Figure 2. $Re = 1000$. Amplification curves associated with the short-term optimals for optimization times $\tau = 1, 2, \ldots, 40$ rotation periods of the vortex. The computations have been carried out as long as $\tau < \tau_{\text{opt}}$, except for $\tau = 20$ where the whole curve is represented for all $k$ (dotted curve). The optimal wavenumber $k_{\text{max}}$ is shown on each curve with an open circle, and the filled circle shows the theoretical limit $G(k=0) = 1$.

Another comment associated with these results is the fact that no amplification can occur in the exact two-dimensional limit. Basically this limit corresponds to a slight modification of the base vorticity profile, and is therefore subjected to the same diffusion mechanism and time scale as the base flow. The mathematical counterpart of this statement is the self-adjointness of the governing operator in this limit, or, alternatively, the cancellation of the production term in the energy equation due to the vanishing of radial velocity (Smith & Montgomery 1995; Pradeep & Hussain 2006). $G(k=0, t=0) = 1$ therefore constitutes an upper bound
for the energy growth \( G(k=0, \tau) \). But this behaviour is associated with a singular limit, as shown in figure 2, which plots the amplification curves obtained with initial conditions maximizing their energy at a fixed time \( \tau < \tau_{\text{opt}} \), or ‘short-term optimals’ as coined by Corbett & Bottaro (2001). It appears clearly that there is selection of a particular wavenumber \( k_{\text{max}} \) corresponding to the largest energy growth at a given time depending on the time of optimization \( \tau \). As \( \tau \) is progressively increased, a drift of this maximum wavenumber to larger wavelengths is observed. But in the same time, the energy growth in the bidimensional limit is still less than 1. Thus, the amplification curve becomes increasingly steep near \( k = 0 \), allowing large energy excursion for large-wavelength structures† while preserving the ‘no growth’ condition in the two-dimensional case.

4.3. Structure of the optimal perturbation

Figure 3 shows the typical evolution of an optimal initial condition. The perturbation is initially composed of a stack of azimuthal velocity streaks located outside the vortex core, in the quasi-potential zone. Interestingly, this distribution of azimuthal velocity streaks is exponentially localized, and therefore does not change the overall circulation budget. As time evolves, the structure of the optimal perturbation evolves in turn (recall that this perturbation is not of modal type) and vortex rings (azimuthal vorticity tori) of alternate signs form and become increasingly stronger. In the evolution depicted figure 3, the intensification of the vortex rolls is at the expense of the initial velocity streaks, which slowly diffuse. By the end of the sequence, the vortex rings contain almost all the energy.

It might be interesting to compare this evolution to the lift-up effect in plane shear flows that transforms initially weak streamwise rolls into powerful streamwise streaks. Conversely, the present optimal initial condition consists of a set of azimuthal (streamwise) velocity streaks that evolves into a stack of strong vortex rings (streamwise rolls). For convenience, we will therefore denote this counterintuitive

† An analogous behaviour in diffusing plane shear flow has been reported in Luchini (1996).
5. The physical mechanism

Two views will be successively adopted in the following. First, based on momentum conservation equations, the interplay between velocity streaks and vortex rings will be made explicit, as well as its time dependence. Secondly, a physical picture of the emergence of vortex rings will be proposed based on angular momentum conservation. Both complementary views will allow further insight into this simple but non-trivial phenomenon.

Localized outside the vortex core, the optimal initial condition evolves in a region that can reasonably be considered as potential. Without base vorticity, the azimuthal velocity perturbation obeys a pure diffusion equation:

$$\frac{\partial u_\theta}{\partial t} = \frac{1}{Re} \Delta u_\theta. \quad (5.1)$$

Similarly, taking the azimuthal projection of the Helmholtz equation for vorticity, we are left with the following evolution equation for the vortex rings:

$$\frac{\partial \omega_\theta}{\partial t} = 2\Omega(r, t) \frac{\partial u_\theta}{\partial z} + \frac{1}{Re} \Delta \omega_\theta. \quad (5.2)$$

Whereas the streaks evolve free of any hydrodynamic feedback, the vortex rings are completely driven by the streaks. From equation (5.2), which could alternatively be interpreted in terms of standard tilting and stretching arguments (Pradeep & Hussain 2006), it can be seen that the vortex rings development follows a linear time dependence in the inviscid limit (except in the bidimensional limit where the expected decay is obvious from (5.1) and (5.2)). Again, this algebraic time dependence, although simple, is completely filtered out with standard modal arguments (Ellingsen & Palm 1975).

A further insight into the amplification mechanism may now be given, taking an alternative view based on angular momentum conservation arguments. The Lamb–Oseen vortex corresponds to an equilibrium where the centrifugal force acting on fluid particles is balanced by the pressure gradient. Introduction of a perturbation disturbs this equilibrium. More precisely, a fluid particle subjected to an azimuthal velocity disturbance $u_\theta$ will drift radially under the action of the local Coriolis force $F_{\text{coriolis}} = 2\Omega(r, t)u_\theta e_r$. Indeed, following Batchelor (1967, §3.2) it can be easily shown that in a frame corotating with the fluid located at $r$, we have

$$\frac{\partial u_r}{\partial t} = -\frac{\partial p}{\partial r} + 2\Omega(r, t)u_\theta + \frac{1}{Re} (\Delta u) \cdot e_r \quad (5.3)$$

where $u_\theta$ represents the same hydrodynamic disturbance as the one in the fixed frame due to axisymmetry, and the term $2\Omega(r, t)u_\theta$ explicitly originates from the Coriolis force. Thus, if located in a high-velocity streak the fluid particle will be expelled radially outwards, while it will be pushed to the centre-axis if the streak is a low-velocity region. Generalizing this reasoning for each fluid particle, it is possible to construct figure 4(a). This alternative view of the initial condition represents the local Coriolis force field acting in the fluid at initial time. Since an azimuthal velocity perturbation is purely diffusing in the potential region, it becomes possible to see the initial stack of streaks as an inhomogeneous distribution of local Coriolis body force, fixed in the bulk and slowly diffusing, with momentum diffusivity.
Figure 4. $k = 1$, $Re = 1000$. (a) ‘Coriolisogram’ of the optimal initial condition and its Helmholtz decomposition into (b) potential and (c) rotational parts.

But so far, the collective behaviour of the particles has been neglected, and the ‘Coriolisogram’ proposed only offers a partial view of the mechanism. Hydrodynamic interactions between particles can be investigated by taking the divergence of the governing equations. This results in the following Poisson equation for the pressure field $p$:

$$\Delta p = \frac{1}{r} \frac{\partial}{\partial r} (2r \Omega(r, t) u_\theta), \quad (5.4)$$

which can be solved numerically, with an appropriate shooting method for example. The signification of equation (5.4) is made clearer if one recognizes the divergence of the Coriolis force on the right-hand side. It should now be evident that the pressure force $-\nabla p$ is the opposite of the Coriolis force’s potential part. Physically, the equivalent body force field in figure 4(a) induces compression and dilation of fluid particles, which are not allowed in the incompressible evolution considered here. Pressure enforces incompressibility by cancelling the potential part of the Coriolis force (figure 4b). As a result, the effective force field acting on fluid particles is the rotational part of the Coriolis force (figure 4c), a result readily visible through the azimuthal vorticity evolution equation (5.2) where the first term of the right-hand side is $\nabla \times F_{\text{coriolis}} \cdot e_\theta$. (Note that in the bidimensional limit, the rotational part of the Coriolis force vanishes.) In summary, the evolution of the optimal perturbation and the underlying mechanism can be explained on the basis of simple physical arguments: the initial streaks of azimuthal velocity generate a distribution of Coriolis force whose rotational part is responsible for the production of azimuthal vorticity and the subsequent emergence of vortex rings.

The physical understanding of the amplification mechanism allows insight into its scaling laws. According to equation (5.1), an azimuthal velocity streak of amplitude $O(\varepsilon)$ stays fixed in the fluid on a time scale $O(Re)$. Vortex rings are slaves of the streaks, as shown with equation (5.2), and encounter an amplification stage bringing them to an amplitude $O(\varepsilon Re)$. As for the lift-up effect, a $Re^2$ scaling for the energy is thus expected. Rescaling the optimal times of figure 1(b), it can be shown that the diffusive time scale is exactly verified. But so far, the scaling predicted for the energy growth appears to be an upper bound. The base vorticity, neglected in the present analysis, is responsible for the alteration of the scaling law (Antkowiak 2005; Pradeep & Hussain 2006). Indeed, the $Re^2$ scaling has been rigorously verified in an unpublished study conducted by the first author and Michel Rieutord in the case of vanishing epicyclic frequency (i.e. potential) Taylor–Couette flow.
6. Conclusions

A powerful mechanism of disturbance energy amplification has been found in vortices. Particular initial conditions that optimally exploit this effect have been exhibited, and their dynamics decrypted. Such optimal disturbances are typically composed of a stack of azimuthal velocity streaks at the initial time. As time evolves, this arrangement of streaks continuously deforms and transforms into a set of contrarotating vortex rings wrapped around the vortex core. The whole process is characterized by a strong amplification of disturbance kinetic energy, even at moderate Reynolds numbers. Moreover this amplification scales as $Re^2$ in the ideal limit of potential base flow, whereas the optimal time follows a diffusive $Re$ time scale. These scalings, as well as the structures involved (longitudinal streaks and rolls), have properties analogous with the lift-up phenomenon occurring in plane shear flows. It must be stressed that these two phenomena act in a reverse fashion, and are based on totally different physical ingredients: whereas a shearwise disturbance (roll) exploits the ambient shear to induce large longitudinal deviations (streaks) in a vortical layer, an azimuthal velocity disturbance (streak) injected in a potential rotating flow exploits the ambient rotation and triggers the development of intense vortex rings (rolls) as a consequence of angular momentum conservation through the generation of a particular distribution of Coriolis force.

This mechanism is generic in two ways. On the one hand, it does not depend on the core vorticity profile, as it is active in the quasi-potential part of the flow. On the other hand, though shown with a particular perturbation profile here, it will be activated by any azimuthal velocity disturbance localized outside the vortex core. In particular, a continuous random forcing should exploit the underlying physical mechanism presented here. Eventually, the result of the optimal perturbation (sometimes termed a ‘pseudomode’) should emerge from the noise, as demonstrated in the case of plane shear flows (Farrell & Ioannou 1993). Obviously, this conjecture has to be confirmed with a proper stochastic forcing analysis, which is currently under way (some former RDT analyses conducted on vortices are already clues of such behaviours, see Miyazaki & Hunt 2000). If confirmed, this scenario would explain the propensity of vortices to develop characteristic vortex ring structures at their periphery when submerged in a disturbed environment such as a turbulent background.

Another result of the present study raises some questions about standard modal stability analyses of vortices. These assume on the one hand that transients have died away (asymptotic long time analysis), but require on the other hand the steadiness of the flow. This latter assumption implicitly restricts the validity in time of such analyses, the diffusive time scale $O(Re)$ being an upper bound. Conversely, the present work, considering the viscous spreading of the vortex, shows that transients may exist, if not arise, precisely on this diffusive time scale. Therefore these results suggest that care should be taken with predictions obtained with modal approaches, at least regarding the evolution of axisymmetric perturbations.

Finally, an interesting mechanism that is undetectable with standard approaches has been shown in vortices, using the optimal perturbation identification as a tool of investigation. This mechanism is specific to rotating flows and is not just a variant of well-known plane shear flows mechanisms. As illustrated here, the optimal perturbation analysis thus provides with a useful framework for revealing original physical mechanisms that exploit the intrinsic properties of the flow: shear or differential rotation, but also potentially stratification, surface tension, etc.
REFERENCES


Stochastic forcing of the Lamb–Oseen vortex

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The aim of the present paper is to analyse the dynamics of the Lamb–Oseen vortex when continuously forced by a random excitation. Stochastic forcing is classically used to mimic external perturbations in realistic configurations, such as variations of atmospheric conditions, weak compressibility effects, wing-generated turbulence injected in aircraft wake, or free-stream turbulence in wind tunnel experiments. The linear response of the Lamb–Oseen vortex to stochastic forcing can be decomposed in relation to the azimuthal symmetry of the perturbation given by the azimuthal wavenumber $m$. In the axisymmetric case $m = 0$, we find that the response is characterised by the generation of vortex rings at the outer periphery of the vortex core. This result is consistent with recurrent observations of such dynamics in the study of vortex-turbulence interaction.

When considering helical perturbations $m = 1$, the response at large axial wavelengths consists of a global translation of the vortex, a feature very similar to the phenomenon of vortex meandering (or wandering) observed experimentally, corresponding to an erratic displacement of the vortex core. At smaller wavelengths, we find that stochastic forcing can excite specific oscillating modes of the Lamb–Oseen vortex. More precisely, damped critical-layer modes can emerge via a resonance mechanism. For perturbations with higher azimuthal wavenumber $m \geq 2$, we find no structure that clearly dominates the response of the vortex.

1. Introduction

From the large hurricanes developing in the atmosphere to the well-known Kelvin-Helmholtz billows in shear layers, vortices are ubiquitous in fluid flows. They are notably major actors of turbulence as they are involved in the energy cascade, entrainment and mixing. The understanding of their dynamics is then of considerable interest. In the context of aeronautics, the will to reduce the aircraft wake and the associated hazards to forthcoming planes has motivated the study of the stability of columnar vortices. Since the early works of Crow (1970), Moore & Saffman (1975) and Tsai & Widnall (1976), many studies have shown the sensitivity of the wing-tip vortex pair to both long- and short-wave cooperative instabilities, see Leweke & Williamson (1998) or Billant et al. (1999) amongst others. Conversely, a single vortex is asymptotically stable but it supports various families of oscillating and damped modes amongst which are the so-called Kelvin waves (Fabre et al. 2006). Although commonly used, the standard modal stability approach fails to fully predict the vortex linear dynamics. Indeed, transient growth can occur when specific perturbations are introduced in the flow. Antkowiak & Brancher (2004) have calculated such disturbances for the Lamb–Oseen vortex and evidenced a core contamination mechanism combining Orr (1907a,b) and induction effects. The associated energy amplification can reach levels high enough to activate the nonlinearities and eventually lead to another equilibrium state or trigger a so-called “bypass” transition.
to turbulence. In the axisymmetric case, Pradeep & Hussain (2006) (cited as PH06 in the following) recently encountered similar transient amplifications. Antkowiak & Brancher (2007) (hereinafter referred to as AB07) completed the picture with the identification of a physical mechanism specific to vortices leading to the generation of vortex rings in the potential region around the vortex core.

The inability of modal analysis to prefigure such transient energy amplification was previously keyed out in wall-bounded shear flows, see Butler & Farrell (1992, 1993), Reddy & Henningson (1993) and Farrell & Ioannou (1993a). Thus both Couette and Poiseuille plane flows experience energy amplification even in their modal stability domain when aroused with the adequate perturbation. The physical mechanism involved consists in the emergence of strong streamwise velocity streaks emanating from streamwise vortices in the flow.

This general occurrence of transient growth in asymptotically stable flows is intimately related to the non-normality of the associated linear dynamical operator (Trefethen et al. 1993; Farrell & Ioannou 1994). Let us consider a flow represented by the following dynamical system for the state vector $x$

$$\frac{dx}{dt} = Ax,$$  \hspace{1cm} (1.1)

where $A$ is the operator corresponding to the Navier-Stokes equations linearised around a given basic state. The base flow is said to be asymptotically stable if all the eigenvalues of $A$ have a negative real part. When the dynamical operator is non-normal – i.e. $A^H A \neq AA^H$ with $(.)^H$ denoting the Hermitian transpose (Farrell & Ioannou 1993b, 1994) – this approach does not address the issue of the flow energetics for finite times. Indeed, the stability analysis of a non-normal operator results in a set of modes that decay individually but that do not form an orthogonal basis. As a result, one can construct a perturbation on this basis with expansion coefficients that are large but with modes cancelling each other to give an initial energy of order one. Since each eigenmode evolves independently, the initial cancellation may not persist. The energy of the disturbance can thus increase substantially before decaying ultimately to zero. Schmid (2007) gives an illustrative two-dimensional geometric example of this scenario in his recent review on nonmodal stability analysis. This mathematical property of non-normal operators is revealing of physical mechanisms that lead to transient energy amplification. The lack of orthogonality corresponds to the potential for energy extraction from the basic flow by a subspace of perturbations leading to transient growth despite the absence of modal (i.e. exponential) instability, a result already pointed out in the seminal work of Orr (1907a,b).

This specificity of non-normal operators can be explained in a more formal way by considering the equation governing the instantaneous rate of energy change

$$\frac{dE}{dt} = \frac{d}{dt} (x^H x) = x^H (A^H + A) x,$$  \hspace{1cm} (1.2)

where the energy of the system is defined by $E = x^H x$ in the $H_2$ norm. Energy growth occurs when the right-hand side of equation (1.2) is positive. This is mathematically equivalent to requiring that a portion of the numerical range of $A$, defined by $N(A) = \{ z \in \mathbb{C}, z = x^H Ax, x^H x = 1 \}$, lies in the right half-plane (Reddy et al. 1993). This can be turned into a condition on the spectrum of the energy operator $K = A^H + A$, since the largest eigenvalue of $\frac{1}{2} K$, referred to as the numerical abscissa (Schmid 2007), is the supremum of Re $[N(A)]$. Transient growth of energy is likely to occur if $K$ has at least one eigenvalue with a positive real part.
For flows experiencing such transient growth, it is of interest to find the “most dangerous initial condition” through a process of optimisation. It consists in identifying the initial condition that maximises its energy growth at a fixed time $\tau$. Such a disturbance is referred to as an optimal perturbation. A second optimisation can be conducted with respect to time, leading to the optimal time $\tau_{\text{opt}}$ that defines the most amplified optimal perturbation, often called the global optimal perturbation. Besides, in some cases, it is also relevant to look at the other disturbances experiencing transient energy amplification. They are termed sub-optimal perturbations. Among them, short-term optimals correspond to optimal perturbations for fixed times $\tau < \tau_{\text{opt}}$ (Corbett & Bottaro 2001).

Given the existence of such optimal disturbances in columnar vortices (Antkowiak & Brancher 2004; Pradeep & Hussain 2006; Antkowiak & Brancher 2007), the point is to know if they can naturally emerge from uncontrolled perturbations as diverse as atmospheric turbulence, background noise in wind tunnel experiments or turbulence generated by the aircraft wings. Indeed, while the potential for substantial transient growth of properly defined initial perturbations certainly exists, recurrent criticisms against optimal perturbation analyses concern the particular structure of these disturbances. These can be quite intricate and unlikely to occur spontaneously in real conditions since there is no apparent mechanism to excite such specific perturbations. This issue is theoretically addressed in the present paper to some extent. The general technique is to linearise the Navier-Stokes equations of small perturbations to a particular mean flow and then to augment these linear dynamics with stochastic forcing, which is uncorrelated in time (i.e., “white noise”) and also possibly uncorrelated in space. This general maintained forcing can be interpreted as a crude way to mimic perturbations arising incessantly in real transitioning flows due to background turbulence or any kind of uncontrolled (in space and time) ambient fluctuations. But it should be kept in mind that the main objective of the present work is to understand the role played by the transiently growing disturbances found in the previous optimal perturbation analyses when the forcing lacks the bias of any specific forcing function. In that context, while this analysis may not necessarily apply directly to, say, real-life turbulent vortical flows where only certain types of disturbances may be introduced, it will nevertheless lend insight into the importance of the physical mechanisms of transient growth uncovered by the previous optimal perturbation analyses.

The associated dynamical equations can be thought of as a system where background noise is regarded as an “input” and the resulting random velocity field representing the response of the flow as the “output”. The ratio of the output energy or variance to that of the input noise gives the energy amplification or gain of the system (Schmid 2007). This analysis was successfully applied to wall-bounded shear flows (Farrell & Ioannou 1993b; Bamieh & Dahleh 2001) and to two-dimensional vortices with radial inflow derived from geophysical applications (Nolan & Farrell 1999). In atmospheric sciences this approach is indeed a classical tool of investigation for the prediction of the statistics of meteorological flows such as the long- and short-term deviations of the hurricane tracks from that prescribed by the surrounding flow. For instance, Whitaker & Sardeshmukh (1998) used such a stochastic forcing analysis to recover successfully the observed variances of the winter Northern Hemisphere flow (in particular the location and structure of the storm tracks). Not only are these previous works in good agreement with the results issued from optimal perturbation analyses, pointing out the robustness of the growth mechanisms uncovered by these latter, but they also show to be complementary to them.

The paper is organised as follows. The stochastic forcing formulation is introduced for any general linear dynamical system in §2. Then its derivation for the Navier-Stokes equations is considered. The results are presented in §3, classically ordered with increasing
azimuthal wavenumbers. The results are discussed and interpreted in §4. The paper ends with the conclusions and perspectives to this work in the last section §5.

2. Numerical formulation

2.1. Stochastically driven linear dynamical systems

The formalism employed in this section is classical in control theory and we only give a brief synthesis of the main steps for the sake of completeness. We consider an asymptotically stable linear dynamical system of the form (1.1) under the influence of an external forcing $\xi$. The system of governing equations can be conveniently written in the following state-space form classically used in the control literature (Jovanovic & Bamieh 2005)

$$\frac{dx}{dt} = Ax + B\xi, \quad (2.1a)$$

$$y = Cx, \quad (2.1b)$$

where $x$ is the state vector with initial condition $x_0$ and $y$ is the output vector. The matrix $B$ and $C$ denote the input and output operators. The forcing considered is stochastic in nature and is assumed to be a temporal Gaussian white noise process with zero mean

$$\langle \xi \rangle = 0, \quad (2.2a)$$

$$\langle \xi(t)\xi^H(t') \rangle = R\delta(t - t'), \quad (2.2b)$$

where $\langle \rangle$ is the ensemble averaging operator and $R$ the spatial covariance matrix. For purpose of representativeness of a specific configuration, the matrix $R$ can be implemented with respect to experimental data. In the absence of such information it can be set equal to the identity matrix, i.e., $R_{ij} = \delta_{ij}$, leading to a spatio-temporal Gaussian white noise. Some refinements may be included in the forcing term in order to represent more specific perturbation fields. For instance, one can let the forcing amplitude peak near the walls to take into account their influence in wind tunnel experiments as done by Jovanovic & Bamieh (2005) for wall-bounded shear flows. We limit here the statistical properties of the forcing terms to the above-mentioned so as to mimic the most generic free stream disturbances occurring in real conditions without favouring any particular region of the flow. As claimed by Farrell & Ioannou (1993b), this forcing is not intended to reproduce the full complexity of turbulence observed in experiments. Its aim is to retain the essential physics underlying the maintained variance from any external continuous perturbation field. In that sense, the analysis will give an insight into the receptivity of the flow without introducing any a priori bias through the characteristics of the forcing in the physical or spectral space.

As a response to this forcing, the emerging state $y$ is a stochastic process with second-order statistics given by the covariance matrix

$$\langle y(t)y^H(t) \rangle = C \int_0^t e^{A(t-s)}BB^He^{A^H(t-s)}ds C^H = CQ(t)C^H, \quad (2.3)$$

where $Q(t)$ is referred to as the controllability Gramian. If the dynamical operator $A$ does not vary in time, the system will reach a statistical steady state where the matrix $Q_\infty = \lim_{t \to \infty} Q(t)$ is solution of a Lyapunov equation

$$AQ_\infty + Q_\infty A^H = -BB^H.$$

(2.4)

The mean energy corresponds to the variance of the output stochastic process and can be extracted from the covariance, i.e. $\langle E(t) \rangle = \langle y^H(t)y(t) \rangle = \text{trace} [B^HQ(t)B]$. As
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classically done in the control literature (Zhou et al. 1995; Schmid 2007), this quantity can be interpreted as the $H_2$-norm of the transfer function $H(\omega) = C(i\omega I - A)^{-1}B$ associated to the linear system (2.1). To obtain the energy amplification denoted as $G_\infty = \langle E_\infty \rangle / \langle E_{inp} \rangle$, one has to determine the input energy introduced by the stochastic forcing. It comes from the variance equation which reads

$$
\frac{\text{d}}{\text{d}t} \langle E(t) \rangle = \langle x^H (A^H C H C + C^H C A) x \rangle + \langle x^H C^H C B \xi \rangle + \langle \xi^H B^H C^H C x \rangle.
$$

Using (2.1b) and (2.2b), the input energy appearing on the left-hand side of (2.5) becomes

$$
\langle E_{inp} \rangle = \langle x^H C^H C B \xi \rangle + \langle \xi^H B^H C^H C x \rangle = 2 \text{trace} [B^H (C^H C)^2 B].
$$

The extraction of coherent structures from the random flow field resulting from the forcing can be obtained through the computation of the eigenmodes of the controllability Gramian. The eigenvalue decomposition of $Q_\infty$, known as the Karhunen-Loève (KL) or proper orthogonal decomposition (POD), will provide with the flow patterns that participate to the response, ordered according to their contribution to the variance of the statistically steady state (Schmid 2007)

$$
Q_\infty \zeta^{(p)} = \gamma^{(p)} \xi^{(p)}.
$$

These distributions will be referred to as output structures hereinafter. It is also of interest to know which coherent structures from the input noise participate the most to the excitation of the system. This issue can be addressed by considering the adjoint dynamical system† $(A^H, B^H, C^H)$ forced with a similar Gaussian white noise. The second-order statistics of the adjoint output stochastic process are contained in the covariance matrix $B^H P(t)B$ where $P(t)$ is the observability Gramian defined as

$$
P(t) = \int_0^t e^{A^H (t-s)} C^H C e^{A(t-s)} ds.
$$

Its long-time value is also solution of a Lyapunov equation

$$
A^H P_\infty + P_\infty A = -C^H C.
$$

Finally, the eigenvalue decomposition of $P_\infty$, coined the back Karhunen-Loève decomposition by Farrell & Ioannou (1993b), will provide with the coherent forcing structures (referred to as input structures in the sequel) hierarchised according to their contribution to the system excitation

$$
P_\infty \zeta^{(p)} = \beta^{(p)} \zeta^{(p)}.
$$

2.2. Stochastic forcing applied to Navier-Stokes equations

The base flow considered in this paper is the Lamb–Oseen vortex. Its non-dimensional azimuthal velocity field is

$$
V(r) = \frac{1}{r} \left( 1 - e^{-r^2} \right),
$$

where the characteristics scales used are the vortex dispersion radius $r_0$ and the angular velocity at the axis $\Omega_0$. In the following, $\Omega(r) = V(r)/r = (1 - e^{-r^2})/r^2$ and $Z(r) = (1/r) \partial_r (r V(r)) = 2 e^{-r^2}$ will represent respectively the angular velocity and the axial

† A specific notation, say $A^+$, should have been used for the adjoint operator as its definition strictly depends on the chosen inner product. In the present case the adjoint operator $A^+$ coincides with the Hermitian transpose $A^H$ since the standard inner product of the Hilbert space $\mathcal{H}$ with identity weighting is used (Hoepffner 2006).
vorticity of the columnar vortex. We now consider infinitesimal disturbances expanded in a modal form

\[ \hat{u}_r, \hat{u}_\theta, \hat{u}_z, \hat{p} \{ (r, \theta, z, t) = [u, v, w, p](r, t)e^{im\theta + ikz} + c.c. , \]  

(2.12)

where \( c.c. \) stands for the complex conjugate, \( m \) is the azimuthal wavenumber and \( k \) the axial wavenumber. The linearisation of the incompressible Navier-Stokes equations gives the following set of equations

\[
\begin{align*}
\frac{1}{r} \partial_r (ru) + \frac{im}{r} v + ikw &= 0, \\
\partial_t u + im\Omega u - 2\Omega v &= -\partial_r p + \frac{1}{Re} \left( \left( \Delta_{m,k} - \frac{1}{r^2} \right) u - \frac{2im}{r^2} v \right), \\
\partial_t v + im\Omega v + Zu &= -\frac{im}{r} p + \frac{1}{Re} \left( \left( \Delta_{m,k} - \frac{1}{r^2} \right) v + \frac{2im}{r^2} u \right), \\
\partial_t w + im\Omega w &= -ikp + \frac{1}{Re} \Delta_{m,k} w,
\end{align*}
\]

(2.13a,b,c,d)

where the Reynolds number is \( Re = \frac{\Omega_0 r_0^2}{\nu} = \frac{\Gamma}{(2\pi \nu)} \) and \( \Delta_{m,k} \equiv \partial_{rr} + \frac{1}{r} \partial_r - \frac{m^2}{r^2} - k^2 \).

The set of variables is reduced to the three velocity components by eliminating the pressure with the Poisson equation. The state variables correspond to the primitive ones, i.e. \( x = [u, v, w] \), and both the input and output operators \( B \) and \( C \) of §2.1 simply reduce to the identity operator. A Chebyshev spectral collocation method with an algebraic mapping identical to that described by Fabre & Jacquin (2004) and Fabre et al. (2006) is used for the spatial discretisation of the problem. The energy of the perturbation, which defines a weighted inner product, is defined classically by

\[ E = (x, x)_E = \frac{1}{2} \int_0^\infty (u^*u + v^*v + w^*w) r dr, \]

(2.15)

where \( * \) denotes the complex conjugate. The use of the energy-based inner product requires a coordinate transform to convert the statistics measures into the more standard Hermitian \( H_2 \) norm (Hoepffner 2006; Schmid 2007). From the Choleski factorisation of the symmetric definite positive weight operator \( W = M^HM \), the relation between the two norms is \( ||x||_E = ||Mx||_2 \) and \( ||A||_E = ||MAM^{-1}||_2 \).

The calculations are carried out with MATLAB using the DMSuite package developed by Weideman & Reddy (2000). The convergence of the numerical procedure depends on both the truncation level \( N \) and the radial domain extent \( r_{max} \). We tested the results sensitivity to both parameters and converged calculations were obtained for \( N \in [150, 300] \) and \( r_{max} \in [5000, 20000] \), according cases.

3. Results

3.1. The axisymmetric case (\( m = 0 \))

The presentation of the results starts with the axisymmetric case. Figure 1 shows the energy amplification as a function of the axial wavenumber for Reynolds numbers varying from 500 to 10000. As in optimal perturbation analyses (PH06, AB07), high levels of amplification are observed. The plot also confirms that the largest amplifications occur for small axial wavenumbers. The amplification levels reach higher values than those found by
optimal perturbation analyses, which can be easily understood since the system is excited by a constant input and not solely by an initial condition (Bamieh & Dahleh 2001). This difference in the level of amplification between these two approaches is systematic.

For the $k = 1$ case, we consider the hierarchy of both the input and output structures according to their contribution to the sustained variance of the steady state. The spectra resulting from the eigenvalue problems (2.7) and (2.10) are plotted in figure 2. The energy amplification essentially results from the contribution of the few first structures as the eigenvalues rapidly decrease to negligible values. The first input structure is responsible for 62% of the vortex excitation and dominates the forcing field. In the same manner, the first output structure prevails in the vortex response by contributing to 61% of the variance sustained by the flow. Figure 3 gives the spatial distribution of these two dominant structures. They are very similar to the optimal perturbation at initial and optimal times found in the optimal perturbation analyses of PH06 and AB07. The forcing structure consists of azimuthal velocity streaks located in the quasi-potential region in the outer periphery of the vortex. The physical mechanism leading to the azimuthal vortex rings of the figure 3(b) has been previously explained by AB07. It has been called “anti-lift-up” in reference to the so-called “lift-up” mechanism occurring in planar shear flows, though it is radically different in nature. Briefly, the azimuthal velocity streaks induce a local Coriolis force field yielding to a radial displacement of the fluid particles. Its potential part is balanced by the pressure gradient to ensure the flow incompressibility. Its rotational part feeds the tori of azimuthal vorticity that constitute the vortex response displayed in figure 3(b). AB07 established this original mechanism with the assumption of zero base vorticity. This is a good approximation since the velocity streaks are localized in the quasi-potential region of the base-flow. The essential trends are correctly captured by this scenario. Nevertheless, the residual base vorticity in the outer periphery of the Lamb–Oseen vortex is at the origin of a small deviation from this idealized view when approaching the vortex core. Indeed, one can identify weak secondary rolls in figure 3(b) located closer to the vortex core. Antkowiak (2005) showed that they are the signature of
waves generated in the region of non-zero epicyclic frequency of the base-flow \( \kappa \) defined by \( \kappa(r)^2 = 2\Omega(r)Z(r) \) (Rayleigh discriminant). Mathematically, this comes from the coupling term associated with the base vorticity \( Z(r) \) in the equation for the azimuthal velocity, see equation (2.13c).

When varying the axial wavenumber, the radial location of both the azimuthal velocity streaks and the resulting azimuthal vortex rings increases with increasing wavelengths (data not shown). This is in agreement with the results of PH06 who related the increasing radial position of the optimal perturbation (and consequently the associated optimal time) with increasing \( k \) to the radial distribution of the vorticity-to-strain ratio in the base flow. The anti-lift-up mechanism is observed to dominate the flow dynamics even more as the axial wavenumber decreases. Indeed, contribution of the dominant structures to the variance of the response is found to shrink at large \( k \). For \( k = 0.1 \), the output (resp. input) structure accounts for 78\% (resp. 79\%) of the energy amplification, whereas
for $k = 2.5$ the output (resp. input) structure only participates for 32% (resp. 32%) of the gain.

The next point to be discussed is the influence of the Reynolds number. From figure 1, it is obvious that larger Reynolds numbers result in larger energy amplifications by the flow. This is not surprising because of the energy balance between the extraction of energy from the background flow by the stochastic forcing, on the one hand, and the viscous dissipation on the other hand. When looking more precisely at the Reynolds number dependence for the sustained variance, one can obtain a scaling law from a sufficient collection of calculations, see figure 4(a) for $k = 1$. For the range of axial wavenumbers explored here, the scaling was always found to be of the form $G_\infty = O(Re^\alpha)$ with $\alpha$ decreasing from 3 for $k = 0.1$ to 2.5 for $k = 3$. Antkowiak (2005) showed that a $O(Re^2)$ scaling for the transient energy growth is an upper bound. In the present study, the system is being continuously excited and the energy accumulates before it dissipates. The characteristic diffusion time is $O(Re)$. Hence, the upper bound for the energy growth is $O(Re^3)$ here, which is consistent with the results obtained. This scaling has been derived with the hypothesis of zero base vorticity. The deviation from the predicted scaling law is explained by the existence of the radially propagating waves mentioned previously, which are responsible of an outward radiation of energy (Antkowiak 2005). This point also enlightens on the variation of the exponent $\alpha$ with $k$ in the scaling law found in this study. As $k$ is decreased, the input structures are located further away from the core where the zero base vorticity assumption is more accurate (PH06, AB07). Hence for small wavenumbers, there is no wave generation and the scaling law in $O(Re^3)$ becomes exact.

In addition, variation of the viscous diffusion does not affect the spatial structure of forcing and response. The velocity and vorticity distributions displayed in figure 3 remain unchanged when the Reynolds number is varied. The only difference resides in the peaks of both azimuthal velocity and vorticity that rise to higher values when the Reynolds number is increased. Finally, we also obtained an empirical scaling law for the energy amplification when $k$ goes to zero. We found numerically that the gain roughly scales as $O(Re^3/k^2)$ as can be seen in figure 4(b).
Figure 5 displays the energy gain for the $m = 1$ helical waves as a function of the axial wavenumber for different Reynolds numbers. Compared to the optimal perturbation analyses of Antkowiak & Brancher (2004) and PH06, similar global trends are retrieved and large amplification levels are found. The large increase of the gain for small $k$ is observed as well as the peak around $k = 1.5$ and an emerging one at $k = 2.5$ although less pronounced than in the optimal perturbation analysis of Antkowiak & Brancher (2004). The latter point suggests that the dominant structure does not emerge conspicuously in the vortex response for this range of wavenumber. This is confirmed quantitatively since the first output structure only contributes to 32% of the sustained variance for $k = 1.35$ and $Re = 1000$. This is the consequence of the coexistence of several perturbations participating in the energy amplification when the system is stochastically forced. For $k = 1.35$, the second emerging structure contributes for 11% to the gain. Hence, the vortex response to the forcing will be dominated by the first output structure. Its axial vorticity is plotted in figure 6 as well as the axial vorticity of the input structure that essentially excites it. This forcing structure is composed of a pair of left-handed spiraling vorticity sheets similar to the initial shape of the optimal perturbation found by Antkowiak & Brancher (2004). These two folded vorticity layers located in the quasi-potential region of the flow are of alternate sign. Their respective velocity induction on the vortex core initially cancel each other. As time evolves, they progressively uncoil via an Orr mechanism induced by the base flow differential rotation. Through this process, the reorganisation of the vorticity sheets promotes an increasing velocity induction in the vortex core that leads to the emergence of the dominant output structure of figure 6(b). This response corresponds to the first critical layer mode of Fabre et al. (2006) as can be seen from the comparison of the velocity radial profiles in figure 7(a).

This phenomenon can be interpreted as a transient resonance mechanism: a Kelvin wave of the Lamb–Oseen vortex is excited by the perturbation field induced by the uncoiling of the initial vorticity spirals. This conjecture was first confirmed by Antkowiak
Figure 6. Isocontours of axial vorticity for the dominant structures for $m = 1$ and $Re = 1000$. The same convention as in figure 3 is used except for (d) where ten equally spaced levels have been used. The dotted circle corresponds to the location of the maximum azimuthal velocity of the Lamb–Oseen vortex at $r = 1.1290 \rho_0$. First input structure for (a) $k = 1.35$ accounting for 31% of the energy amplification, (c) $k = 2.5$ and 17%, (e) $k = 0.5$ and 50%. First output structure for (b) $k = 1.35$ accounting for 32% of the flow excitation, (d) $k = 2.5$ and 16%, (f) $k = 0.5$ and 52%.
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Figure 7. (a) Comparison of the velocity radial profiles between the first L1 mode of Fabre et al. (2006) and the output structure emerging from the stochastic forcing for $k = 1.35$ and $Re = 1000$. The solid, dashed and dotted lines correspond respectively to the radial, azimuthal and axial velocity of the L1 wave. The cross markers denote the velocity profiles of the output structure. (b) Influence of the Reynolds number on the energy amplification factor for $k = 1$ and $m = 1$. The cross markers represent calculations and the dashed-dotted line the theoretical scaling derived from Antkowiak (2005) analysis. For this axial wavenumber, the linear regression from the numerical data gives a gain scaling as $G_\infty = \mathcal{O}(Re^{1.8})$.

(2005) who made a successful comparison with a forced harmonic oscillator elementary model. Choosing an adequate forcing term so as to correctly reproduce the influence of the spiraling vorticity arms, the frequency of the forcing term is found to fit very well those of the selected waves. PH06 performed a similar analysis on the top-hat (Rankine) vortex. Their results supported the hypothesis of a resonance-driven excitation of vortex waves. From the simple harmonic oscillator model, one can obtain an expression for the energy gain at large time when the forcing term has a constant amplitude: $G_\infty = 1/(s^2 + (\omega_f - \omega)^2)$ where $\omega_f$ is the pulsation of the forcing term, $\omega$ the pulsation of an eigenmode of the system and $s$ its damping rate. From this first estimation of the energy growth, one can see that the selection process is based on two aspects. The first criterion consists in selecting one of the least damped waves – minimising the $s$ term – since it allows a maximal energy amplification. For the present set of parameters, the first critical layer mode – referred to as the first L1 mode according to the nomenclature of Fabre et al. (2006) – has the minimum damping rate, see their figure 6. As a consequence it is found to be the dominant output structure. Secondly, the disturbance may be in phase with the wave in order to impose a continuous excitation – minimizing the $(\omega_f - \omega)$ term. This implies that the rotation rate of the spirals be close to the pulsation of the mode. In the quasi-potential region of the flow, the rotation rate of the vorticity sheets is $\Omega(r) \approx 1/r^2$. Hence, the radial position of the spiraling vorticity sheets is imposed by the mode frequency, i.e. $r \approx 1/\sqrt{\omega}$ where $\omega$ is the wave pulsation, a criterion previously established by PH06 with the top-hat vortex model (see their figure 21c). For the $k = 1.35$ case, the first L1 wave frequency is 0.121, which gives a mean radial position of $r = 2.9$. This is consistent with the location of the spiral arms in figure 6(a).

Keeping in mind these two criteria for the resonance phenomenon and considering the stability results of Fabre et al. (2006), it is expected that the mode selection depends on the axial wavenumber. Indeed for larger wavenumbers, we found the second L1 mode as the first output structure, see figure 6(d) for $k = 2.5$. When $k$ is decreased, the emerging structure displayed in figure 6(f) is also of a different nature and corresponds to the displacement wave of the Lamb–Oseen vortex (referred to as the D wave in the
According to the criteria previously mentioned, the emergence of the D mode is not linked to an exact resonance. This wave being countergrade, i.e., its pulsation is negative, no disturbance can have a similar rotation rate. In this case, the D wave emerges only because its damping rate is very small compared to those of the waves of the L family. It is noteworthy that the spiraling input structure is located far from the vortex core, at a mean radial position $r \approx 15$. This represents a slow time-rotating disturbance with an associated rotation rate $\Omega(r) \approx 0.0044$. This point is linked to the fact that the displacement wave emergence corresponds to an exact resonance in the two-dimensional limit for a steady forcing. For $k = 0$, the D wave is stationary and the exciting perturbation can satisfy the in-phase condition if the spiral arms of vorticity are pushed outwards to an infinite radial position. Obviously, this could not be verified numerically, but we found that the radial location of the input structure increased when reducing the axial wavenumber from $k = 0.5$ to $k = 0.1$. This exact resonance comes with an infinite energy growth as revealed by the large increase of the gain when $k$ goes to zero in figure 5.

The emergence of the displacement wave under continuous external perturbations is a good candidate for explaining the vortex meandering (or wandering) phenomenon. The nature of the wave, namely a long wavelength bending wave, is in agreement with the characteristics of the meandering observed in wind tunnel experiments (Baker et al. 1974; Devenport et al. 1996). The theoretical approach used here, which takes into account the external disturbances in the form of a continuous random forcing, is consistent with real experimental conditions where background noise is constantly exciting the wing-tip vortex. This point has to be confirmed experimentally by controlling, or at least, quantifying, the perturbation field and by correlating the data with measurements of the vortex response. Here we only give a physical mechanism that could explain the occurrence of this phenomenon. But we are still far from a comprehensive and fully predictive description that could help experimentalists to filter out this erratic core displacement. To our knowledge, the best way to proceed is proposed by Devenport et al. (1996), who used a Gaussian model for the fluctuations of the vortex core position.

Inspection of the effect of viscosity has been explored numerically and a general power law has been obtained for the energy amplification, see figure 7(b) for $k = 1$. For the range of axial wavenumbers considered here, we found the scaling $G_\infty = O(Re^{1.8})$. This exponent is considered as valid for any $k$ since very small variations were observed, i.e., $1.8 \pm 0.02$. Antkowiak (2005) showed that the transient energy growth is $O(Re)$ for the resonance phenomenon while it is $O(Re^{2/3})$ when only the Orr mechanism is active. Following the same reasoning as in the axisymmetric case, the theoretical upper bound for energy amplification under stochastic forcing is $O(Re^2)$. The results obtained here are in good agreement with this scaling law. A decrease in viscous diffusion also affects the structures. While the vorticity distribution of the Kelvin waves remains practically unchanged, the spiral arms of the dominant disturbance get thinner as the Reynolds number increases, a point already noticed by PH06. The peaks of vorticity also reach larger values, which leads to a stronger velocity induction in the core when the spirals unfold and a higher level of energy amplification.

### 3.3. The double-helix case ($m = 2$) and higher azimuthal wavenumbers

The energy gain of the vortex in response to double-helix perturbations is plotted versus the axial wavenumber in figure 8 for different Reynolds numbers. The levels of amplification are lower than in the axisymmetric case but with similar tendencies. For a given Reynolds number, the larger the wavelength the larger the amplification. These results are noticeably different from those of the optimal perturbation analysis (PH06). Not
only are the levels much higher but the dependence on $k$ is completely different. The amplification maximum occurs for small wavenumbers and the curve does not present the flat-shaped aspect displayed in figure 19(b) of PH06. The difference in the results can be explained by two complementary arguments. Firstly, the development of the optimal perturbation is completed within few rotation periods of the vortex (Antkowiak 2005). In this case, the transient energy growth of the optimal perturbation may occurs on too short a timescale, compared to the statistical mean time between two consecutive excitations of the mode by the external forcing, for being efficiently amplified. After an excitation of the vortex with the optimal disturbance extracted from the random noise, the process of energy amplification and decay is likely to be over before another excitation happens. Hence, the energy gain resulting from successive excitations is statistically unlikely to cumulate. As a result, the optimal perturbation will be less amplified under a random forcing than a disturbance associated with a less efficient transient mechanism but occurring on a longer timescale. This point is related to the second argument. Indeed, strong sub-optimal perturbations competing with the optimal one may exist. They can eventually dominate the response of the vortex when submitted to stochastic maintained forcing if they develop on a larger timescale than the optimal one. Evidence to this conjecture can be obtained by looking at both the input and output structures.

The structures that dominate the input forcing consist of the same kind of spiralling structures as the ones identified in the $m = 1$ helical case (fig. 9). The resonance mechanism leading to the selection of a Kelvin wave is active and the first output structure for $k = 2.5$ displayed in figure 9(b) corresponds to the flattening wave of Fabre et al. (2006). This mode being by far the least damped, it is selected on frequency arguments according to the criteria mentioned in the previous section. The mean radial position of the entangled vorticity spirals is imposed by the pulsation of the mode. However, while active, the resonance no longer dominates the flow dynamics since the first input structure account for only 8% of the sustained variance in this case. The sub-optimals participate equally to the excitation of the vortex. For smaller wavenumbers, transient
Figure 9. Isocontours of axial vorticity for the first functions for $m = 2$ and $Re = 1000$. The same convention as in figure 3 is used. First input structure for (a) $k = 2.5$ accounting for 8% of the energy amplification, (c) $k = 1$ and 10%. First output structure for (b) $k = 2.5$ accounting for 8% of the flow excitation, (d) $k = 1$ and 10%.

resonance is not even the dominant mechanism leading to energy amplification. Figure 9(d) shows the first output structure for $k = 1$. The vortex response is only composed of the four initial vorticity sheets that have been uncoiled. In this case, the Orr mechanism gives the largest energy growth and accounts for 10% of the total amplification. The spiralling vorticity arms of the input structure are located far from the vortex core so that the timescale of the Orr mechanism is long enough to be efficiently amplified by the stochastic forcing. Its timescale is given by the duration of the uncoiling of the vorticity arms which is roughly evaluated by

$$
\tau_{Orr} \approx -\frac{2\pi}{\epsilon \partial_r \Omega(r_h)},
$$

where $\epsilon$ and $r_h$ are respectively the mean width and the radial position of the spiral arm. For the perturbation plotted in figure 9(c), $\tau_{Orr}$ is found to be about 50 rotation periods of the vortex, which is one order of magnitude larger than the optimal time. As
previously argued, this energy growth mechanism is activated on a longer timescale than
the transient resonance, and it is more efficiently amplified by the maintained forcing.
Inspection of the less dominant structures reveals that the flattening wave is the fourth
preferred response of the flow accounting only for 4% of the variance. For the $m = 2$
case, there is no structure that clearly dominates the flow response, by contrast to what
has been found for the $m = 0$ and $m = 1$ cases. Whatever the axial wavenumber, the
contribution of the most dominant output structure never exceeds 15%.

Finally, our survey ends with the description of the vortex response for higher azimuthal
wavenumbers. We explored the flow behaviour under a continuous stochastic forcing for
$m$ as large as 10 and always found energy amplification. The observed gain levels decrease
with increasing $m$ and are far below those of the three previous cases. The amplification
curves present the same shape as for the $m = 2$ case with a maximum when $k$ tends
to zero. The dynamics leading to energy growth always consists of the Orr mechanism.
The dominant input structure lies in the spiralling vorticity sheets that uncoil due to
the differential rotation. No transient resonance mechanism has been found as could
have been expected considering the large damping rate of the Kelvin waves (Fabre et al.
2006). This phenomenon is valid for all $m > 2$. While energy growth is observed, these
cases are of limited interest as they are exceedingly unlikely to prevail over the responses
corresponding to the first three azimuthal wavenumbers.

4. Discussion

The results obtained in the present study show a noticeable similarity between the am-
plification curves for the different azimuthal wavenumbers considered here ($m \in [0, 10]$):
the energy gain is maximum for $k = 0$ and decreases as $k$ increases. This theoretical lack
of intrinsic axial wavelength selection raises questions regarding the use of the present
results in order to predict the response of the vortex in real-life conditions.

At that stage, we wish to underline that the interpretation of the amplification curves
must be undertaken carefully. If one wants to predict the selection of a particular wave-
length, a direct and hasty extrapolation of the results presented in this paper can prove
completely erroneous when compared to experiments without any caution. Thus, con-
sidering that the most amplified $(k, m)$ modes correspond to $m = 0$ and $k$ tending to
zero, it could be tempting to predict the systematic and dominant occurrence of very
large axisymmetric vortex rings at the outer periphery of the vortex, as presented in sec-
tion 3.1. But this conclusion, if applied to realistic experiments or observations, must be
balanced by taking into account the departure of the experimental conditions from the
hypothesis that define the framework of the present analysis. It concerns the unbounded
character of the flow, the spectral content of the external forcing and the influence of the
nonlinearities. These points are discussed further in this section.

4.1. Bounded vs unbounded flows

First, the largest amplifications are obtained in the limit of infinite wavelengths ($k \to 0$)
for every azimuthal wavenumber $m$, suggesting that the response of the vortex should be
systematically quasi-two-dimensional, at least for $m > 0$, and involve very large struc-
tures. But it must be kept in mind that the input structures triggering the vortex response
(azimuthal streaks for $m = 0$ and axial vorticity spirals for $m > 0$) are localised radially
further away from the vortex axis when $k$ decreases, eventually extending to infinity for
$k \to 0$. This singular behaviour questions the validity of the unbounded flow hypothesis
that is implicitly made in the present formulation. It is expected that the predictions
concerning the long-wave response of the vortex put forward in the previous sections will
be distorted by the presence of physical boundaries at finite distance such as the side walls of the wind tunnel, or by taking into account flow features ignored in the present model such as other vortices for instance. The present study found no intrinsic wavelength selection by the vortex and the process of wavelength selection is extrinsic and therefore case-dependent.

Nevertheless, the present results show that the physical mechanisms of growth involved in the response of the vortex systematically favour the structures with the largest axial wavelength admissible as long as their radial extent does not exceed the limit of representativeness of the vortex flow model used here. We then expect the present results to be relevant above a critical axial wavenumber $k_c \sim 2\pi/r_c$ corresponding to a characteristic radial extent $r_c$ above which the vortex flow model used here significantly departs from the real-life conditions.

This discussion can also be supplemented, to a lesser extent, with the validity of the infinite time limit required in the present formulation to reach theoretically a statistical steady state. The input structures that are favoured in the $k \to 0$ limit correspond to infinitely slowly growing disturbances (Pradeep & Hussain 2006; Antkowiak & Brancher 2004, 2007). These perturbations thus may not have enough time to grow significantly in finite-time experiments. This is a second source of distortion for the large-wavelength results presented here. If the flow develops on a short or medium timescale, one can expect the selection of finite-time optimal perturbations which can be radically different from the structures predicted here for an infinite time horizon.

4.2. Initial turbulence vs. continuous white noise

The linear evolution of perturbations of a vortex flow is formally given by the general solution of the governing equation (2.1a) which has two components: the homogeneous solution which describes the evolution of initial conditions and the particular solution which represents the long-time response of the flow to a continuous external forcing. As mentioned by Schmid (2007), both parts are complementary and fully describe the general dynamics of small perturbations.

The studies of perturbed vortices generally focus on the former point, the evolution of initial conditions, such as temporal modal stability analyses (see Fabre et al. (2006) for the Lamb–Oseen vortex), optimal perturbation analyses (Antkowiak & Brancher 2004, 2007; Pradeep & Hussain 2006) or theoretical and numerical investigations of the response of vortices to initially injected turbulence (Melander & Hussain 1993; Risso et al. 1997; Miyazaki & Hunt 2000; Takahashi et al. 2005; Marshall & Beninati 2005). More particularly in the latter case, the objective is to understand how the vortex immersed in an initial turbulent field responds to this perturbation and how in return the initial turbulence is affected by the presence of the vortex and the associated shear and rotation which are known to drastically alter the statistics of turbulence on a short time scale.

By contrast with these studies, the approach adopted in the present paper is quite different, as it focuses on the long-time response of the vortex when submitted to a continuous external forcing. This corresponds formally to the study of the particular solution of equation (2.1a) as stated by Schmid (2007). Physically this particular solution provides a model for receptivity process, where the external forcing may represent either free-stream turbulence, wall roughness or other non smooth geometries, body forces or even neglected terms such as nonlinearities. This forcing can also be linked to the deviation from the model base flow, like the faraway presence of other vortices or the boundary-layer perturbations generated at the side walls of the wind tunnel. In that context of receptivity analysis, the continuous forcing used in the study is chosen as generic and unbiased as possible in the form of white noise. Being equally distributed
in space allows to excite all possible regions of the flow and not to favour any specific wavelength or frequency. Thus, if the wavelength selection is case-dependent (on the dominant wavelengths or frequencies of the external forcing in real-life conditions for instance), see preceding section), the quantification of the amplification is nevertheless a measure of the intrinsic transfer function of the Lamb–Oseen vortex, and reveals the intrinsic mechanisms that are favoured by the flow. Here it is shown that the largest response is observed in forcing scenarios that convert perturbations in the form of azimuthal velocity streaks into intense azimuthal vortex rings (for the axisymmetric part of the flow), and perturbations in the form of spiraling axial vorticity sheets outside the vortex into large bending ($m = 1$) and deformation waves ($m > 1$) within the vortex. It is noteworthy that these mechanisms are similar to the ones uncovered by nonmodal studies of the initial-condition problem such as optimal perturbation analyses (PH06, AB07) and are consistent with the numerical simulations of the interaction of a vortex with an initial turbulent field. This suggests that such mechanisms are robust and fundamental to the dynamics of perturbed vortices. It is expected that they are potentially active whatever the details of the vortex flow and the perturbations considered.

4.3. Validity of the linear approach

This work has been conducted within the linear approximation and it is important to figure out how the results obtained here are adjusted by the nonlinearities. The physical mechanisms of transient energy amplification considered here and in the studies of PH06 and AB07 are linear. As argued by Miyazaki & Hunt (2000), linear analysis correctly describes these processes or at least their initial development, while the nonlinearities impact the further evolution of individual structures. This point is currently under investigation via direct numerical simulations of the nonlinear evolution of the optimal perturbations. Preliminary results for the $m = 0$ case show that the azimuthal vorticity rings of the output structure are self-advected away from the vortex axis, carrying along the streaks of high azimuthal velocity of the forcing structure. Due to the conservation of angular momentum, the intensity of the streaks decreases as the structures move radially outwards. Thus, the axisymmetric mechanism of energy growth is progressively damped by the nonlinearities. In the case of a maintained forcing, this nonlinear damping effect is expected to be of minor importance as the streaks of azimuthal velocity would be statistically regenerated continuously.

Concerning the external forcing from a more general perturbation field, the nonlinear development of the flow has to be considered for all wavenumbers simultaneously. Numerical simulations including a continuous forcing would be of great interest in order to know how the amplification factors obtained here are affected by the nonlinear effects (i.e., saturation). In the case of a transient forcing induced by an initially homogeneous isotropic turbulence, Takahashi et al. (2005) performed such analysis. The departure from the linear regime happens when the bending wave growing in the core reaches a finite amplitude. They measured quantitatively the impact on nonlinearities through the time dependence of the axisymmetric axial correlation function. Its temporal growth was observed to be proportional to $t$ contrary to linear RDT analyses where a quadratic time

† More precisely, as mentioned by one referee, a forcing of the form of a realistic turbulent signal would inject very low amplitude perturbations at large scales $k \to 0$, and therefore a peak is expected at some finite wavelength in the response variance for each $m$. This wavelength selection will depend on the spectral content of the azimuthal Fourier decomposition of the turbulent signal, that might be highly sensitive to the particularities of the experimental environment in the long wave limit.
dependence was found (Miyazaki & Hunt 2000). Thus, the nonlinear terms tend to limit the energy growth and a similar saturation is expected for any generic random forcing.

5. Conclusions

The goal of the present study is to analyse the dynamics of the Lamb–Oseen vortex when continuously forced by a random excitation. Considering the existence of transient growth in vortices when submitted to specific perturbations, this work aims at determining if the optimal perturbations found by Antkowiak & Brancher (2004, 2007) and Pradeep & Hussain (2006) could naturally emerge from background noise. For this purpose, the linear Navier-Stokes equations are continuously forced with Gaussian white noise so as to mimic any perturbations occurring in real transitioning flows. We then look at the large-time statistical response of the vortex and at the coherent structures participating to its excitation. Not only this method allows the energy gain of the system to be quantified but also to find and order the coherent input (resp. output) structures according to their contribution to the excitation (resp. variance or response) of the flow.

For all azimuthal wavenumbers investigated, energy amplification is always observed but when \( m > 2 \), the levels reached are too small compared to those obtained from smaller values of \( m \) to be significant. Compared to the optimal perturbation analysis, the levels of amplification obtained here are always higher since it corresponds to energy growth resulting from a continuous noise input and not from a single initial condition. This difference in the value of the gain is particularly marked when the optimal perturbation evolves on a short timescale. In such cases, the sub-optimal perturbations have to be considered as they participate equally in the vortex response. This point has to be recognised as one advantage of the stochastic forcing approach. Indeed, the optimal perturbation analysis classically focuses on the most amplified disturbance and can sometimes miss some transitional scenarios. This is what we observe for the \( m = 2 \) case where no prevailing mechanism has been found to dominate the vortex dynamics.

Focusing on the physical mechanisms leading to energy growth, the scenarios leading to the vortex excitation identified by Antkowiak & Brancher (2004, 2007) and Pradeep & Hussain (2006) are recovered. This study confirms that the optimal perturbations can naturally be activated by the background noise present in uncontrolled conditions.

In the axisymmetric case, the mechanism called “anti-lift-up” by AB07 consists in the emergence at the vortex periphery of strong tori of azimuthal vorticity fed by the azimuthal velocity streaks of the forcing structure. This scenario gives a theoretical counterpart to the observation of the recurrent development of vortex rings at the periphery of vortices when submerged in an ambient turbulence (Melander & Hussain 1993; Risso et al. 1997; Takahashi et al. 2005).

For helical perturbations, the excitation of the vortex is explained by a transient resonance phenomenon. The dominant forcing structure is composed of two left-handed entangled vorticity sheets located in the quasi-potential region of the flow. As time evolves, they progressively uncoil under the base flow differential rotation and trigger the appearance of a Kelvin wave through a process combining induction effects and the Orr mechanism. The emerging Kelvin mode depends on the axial wavenumber as the selection is based both on a minimisation of the wave damping rate and a concordance between the pulsations of both the mode and the vorticity spirals. At large wavelengths where the gain is maximal, the displacement mode of Fabre et al. (2006) is preferentially excited. The emergence of the displacement wave under the influence of a maintained background noise is thus an interesting candidate for the vortex meandering observed in the experiments (Baker et al. 1974; Devenport et al. 1996).
This mechanism of resonance originating from a localised spiraling vorticity disturbance also occurs for \( m = 2 \) perturbations with the emergence of the flattening wave. But it is in competition with other disturbances experiencing transient growth. The sole disentanglement of the vorticity spirals via the base flow differential rotation leads to energy amplifications as large as those deriving from the flattening mode resonance. The existence of sub-optimal perturbations living on a longer timescale than the optimal one is put forward to explain the discrepancy between the results obtained here and those of PH06. This point has to be confirmed. It will be investigated in the near future with the computation of the sub-optimal perturbations of the Lamb–Oseen vortex for \( m = 2 \).

For larger wavenumbers, the Orr mechanism becomes the dominant process of energy amplification from a continuous random excitation of the vortex.

Finally, the approach conducted here fulfills the interrogations brought by the results of the optimal perturbation studies of Antkowiak & Brancher (2004, 2007) and Pradeep & Hussain (2006). This work demonstrates both analyses to be complementary. Such tools may be applied to asymptotically stable flows whose dynamical operator is non-normal in order to find out an eventual transient mechanism. For example, Joly et al. (2005) recently showed that low-density vortices were insensitive to the Rayleigh-Taylor instability. Since the corresponding dynamical operator does not commute with its Hermitian transpose, nonmodal stability analysis (both initial value and stochastic forcing formulations) should be conducted.

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Spatial stability and the onset of absolute instability of Batchelor’s vortex for high swirl numbers

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Batchelor’s vortex has been commonly used in the past as a model for aircraft trailing vortices. Using a temporal stability analysis, new viscous unstable modes have been found for the high swirl numbers of interest in actual large-aircraft vortices. We look here for these unstable viscous modes occurring at large swirl numbers ($q > 1.5$), and large Reynolds numbers ($Re > 10^3$), using a spatial stability analysis, thus characterizing the frequencies at which these modes become convectively unstable for different values of $q$, $Re$, and for different intensities of the uniform axial flow. We consider both jet-like and wake-like Batchelor’s vortices, and are able to analyse the stability for $Re$ as high as $10^8$. We also characterize the frequencies and the swirl numbers for the onset of absolute instabilities of these unstable viscous modes for large $q$.

1. Introduction

We consider here the spatial stability of the so-called $q$-vortex, also called Batchelor’s vortex, whose velocity field ($U$, $V$, $W$), in cylindrical polar coordinates ($r$, $\theta$, $z$), is given, in dimensionless form (see §2 for more details), by

$$U = 0, \quad V = \frac{q}{r} \left(1 - e^{-r^2}\right), \quad W = W_0 + e^{-r^2},$$

(1.1)

where $q$ is the swirl parameter, and $W_0$ is a uniform axial flow, which can be positive (jet-like vortex), negative (wake-like vortex), or zero. All the velocities are made dimensionless with a characteristic axial velocity $W_c$ that accounts for the exponential part of the axial flow in (1.1). This model vortex is a parallel-flow approximation of the original Batchelor’s (1964) vortex, and has been traditionally used as a simple model for trailing vortices with axial flow (see, e.g. Lessen, Singh & Paillet; Mayer & Powell 1992).

The temporal stability of the $q$-vortex (1.1) was first considered by Lessen and colleagues, both from an inviscid point of view (Lessen et al. 1974), and taking into account the effects of viscosity (Lessen & Paillet 1974). In these works, as in all the temporal stability results of later works commented on below, the uniform advection velocity $W_0$ was set to zero without loss of generality, for this parameter is not relevant in the temporal stability problem because of Galilean invariance. These authors found that (1.1) was unstable to non-axisymmetric counter-rotating perturbations ($n < 0$) in a wide range of the swirl number $q$ and the Reynolds number $Re$ (see §2 for its exact definition). In fact, Lessen et al. (1974) and Lessen & Paillet (1974) only reported inertial instabilities, finding that the widest unstable range of swirl numbers is $0 \leq q \leq q_{\text{crit}} \approx 1.5$, corresponding to $Re \to \infty$ for the azimuthal
wavenumber $n = -1$. These results were later refined numerically by Duck & Foster (1980), and generalized asymptotically for $|n| \gg 1$ by Leibovich & Stewartson (1983), Stewartson & Capell (1985), and Stewartson & Leibovich (1987).

Purely viscous modes corresponding to azimuthal wavenumbers $n = 0$ and $n = +1$ were found by Khorrami (1991a) and by Duck & Khorrami (1992). These modes occur for $q < 1.3$, and their growth rates are always several orders of magnitude smaller than those of the corresponding inertial modes. A detailed characterization of the temporal stability of the q-vortex in the $(Re, q)$-parameter plane, including all the inviscid and viscous modes found previously and some new ones (particularly for $n \geq 0$), was made by Mayer & Powell (1992) (see also the review by Ash & Khorrami 1995).

Viscous instabilities for large swirl numbers ($q > 1.5$) and large Reynolds numbers ($Re > 10^3$) were found asymptotically by Stewartson & Brown (1985). Fabre & Jacquin (2004) made a detailed characterization of these large swirl number viscous instabilities by solving numerically the temporal stability problem. These instabilities were not found previously, particularly in the detailed work of Mayer & Powell (1992), because the unstable perturbations are centre modes which are concentrated along the vortex axis, and they can only be found with a highly accurate spectral method such as that used by Fabre & Jacquin. These authors were able to map the unstable region up to $Re \approx 10^6$ and $q \approx 7$. These unstable modes are related to the family of viscous modes found by Stewartson, Ng & Brown (1988) for a swirling pipe Poiseuille flow.

A spatial stability analysis of the q-vortex, characterizing the unstable regions and the onset of absolute instabilities in the $(Re, q)$-parameter plane for all possible values of $W_0$, was undertaken by Olendraru & Sellier (2002). Although these authors also found unstable modes for large $q$ (up to $q \approx 3$), they did not explore systematically these viscous modes for large $q$. Therefore, the main objective of this work is to perform such a systematic characterization of viscous unstable modes for large values of the swirl parameter and for large Reynolds numbers from a spatial stability analysis. Our numerical technique allows us to reach Reynolds numbers up to the order of $10^8$, with the corresponding swirl numbers up to the order of 100. We first present the results for $W_0 = 0$, and then for $W_0 \neq 0$. In the latter case, the onset of absolute instabilities will also be characterized (no absolute instabilities are found for viscous modes with $q \geq 1.5$ when $W_0 = 0$).

2. Formulation of the problem and numerical method

2.1. General basic flow and spatial stability formulation

Although we shall consider only the spatial stability of the q-vortex (1.1), we formulate here the problem for a general vortex with axial flow which, in cylindrical-polar coordinates $(r, \theta, z)$, and in the parallel-flow approximation, has a velocity and pressure fields given by

$$
U = 0, \quad V = V(r), \quad W = W(r), \quad P = P(r).
$$

(2.1)

All the magnitudes are dimensionless. The flow is characterized by two non-dimensional parameters: a Reynolds number

$$
Re = \frac{r_c W_c}{\nu},
$$

(2.2)

where $r_c$ and $W_c$ are a characteristic radius (the dispersion radius of vorticity) and a characteristic axial velocity, respectively, and $\nu$ is the kinematic viscosity; and a swirl
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parameter,

\[ S_w = \frac{V_c}{W_c}, \]  \hspace{1cm} (2.3)

where \( V_c \) is a characteristic azimuthal velocity. In the case of the Batchelor's vortex (1.1), \( S_w \equiv q \).

To analyse the linear stability of the above basic flow, the velocity \((u, v, w)\) and pressure \(p\) fields, are decomposed, as usual, into their mean parts (2.1), and small perturbations \((u', v', w', p')\). These perturbations are decomposed in the standard form:

\[ s = [u', v', w', p']^T = Se^{az+i(n\theta-\omega t)}, \]  \hspace{1cm} (2.4)

where the complex amplitude,

\[ S(r) = \begin{pmatrix} F(r) \\ G(r) \\ H(r) \\ \Pi(r) \end{pmatrix}, \]  \hspace{1cm} (2.5)

depends only on the radial coordinate in the parallel-flow approximation. The non-dimensional order-of-unity complex radial wavenumber \( a \) is defined as

\[ a = \gamma + i\alpha. \]  \hspace{1cm} (2.6)

The real part \( \gamma \) is the exponential growth rate, and the imaginary part \( \alpha \) is the axial wavenumber. A non-dimensional frequency \( \omega \) has also been defined. Finally, the azimuthal wavenumber \( n \) is equal to zero for axisymmetric perturbations, and different from zero for non-axisymmetric perturbations.

Substituting (2.4)–(2.5) into the incompressible Navier–Stokes equations, and neglecting second-order terms in the small perturbations, the following set of linear stability equations results:

\[ \mathbf{L} \cdot \mathbf{s} = 0, \]  \hspace{1cm} (2.7)

\[ \mathbf{L} = \mathbf{L}_1 + a\mathbf{L}_2 + \frac{1}{Re}\mathbf{L}_3 + \frac{a^2}{Re}\mathbf{L}_4, \]  \hspace{1cm} (2.8)

\[ \mathbf{L}_1 = \begin{pmatrix} \frac{d}{dr} + \frac{1}{r} & i\frac{n}{r} & 0 & 0 \\ i\left(\frac{nV}{r} - \omega\right) & -\frac{2V}{r} & 0 & \frac{d}{dr} \\ \frac{dV}{dr} + \frac{V}{r} & i\left(\frac{nV}{r} - \omega\right) & 0 & i\frac{n}{r} \\ \frac{dW}{dr} & 0 & i\left(\frac{nV}{r} - \omega\right) & 0 \end{pmatrix}, \]  \hspace{1cm} (2.9)

\[ \mathbf{L}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 1 \end{pmatrix}, \quad \mathbf{L}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]  \hspace{1cm} (2.10)
These equations have to be solved with the following boundary conditions:

\[ r \to \infty, \quad F = G = H = 0; \quad (2.13) \]

\[ r = 0 \quad \text{(Batchelor & Gill 1962),} \]

\[ F = G = 0, \quad dH/dr = 0, \quad (n = 0), \quad (2.14) \]

\[ F \pm iG = 0, \quad dF/dr = 0, \quad H = 0 \quad (n = \pm 1), \quad (2.15) \]

\[ F = G = H = 0 \quad (|n| > 1). \quad (2.16) \]

In the spatial stability analysis that will be carried out here, for a given real frequency \( \omega \), and given the parameters \( Re, S_w = q \) and \( n \), the system (2.7)–(2.16) constitutes a nonlinear eigenvalue problem for the complex eigenvalue \( a \). The flow is considered unstable when the disturbance grows with \( z \); i.e. when the real part of the eigenvalue, \( \gamma \), is positive (provided that the group velocity is also positive; see below).

### 2.2. Numerical method

To solve (2.7)–(2.16) numerically, \( S \) is discretized using a staggered Chebyshev spectral collocation technique developed by Khorrami (1991b), where the three velocity components and the three momentum equations are discretized at the grid collocation points, whereas the pressure and the continuity equation are enforced at the mid-grid points. This method has the advantage of eliminating the need for two artificial pressure boundary conditions at \( r = 0 \) and \( r \to \infty \), which are not included in (2.13)–(2.16). To implement the spectral numerical method, (2.7) is discretized by expanding \( S \) in terms of a truncated Chebyshev series. The boundary conditions at infinity are applied at a truncated radial distance \( r_{\text{max}} \), chosen large enough to ensure that the results do not depend on that truncated distance. To map the interval \( 0 \leq r \leq r_{\text{max}} \) into the Chebyshev polynomials domain \(-1 \leq s \leq 1\), we use the transformation

\[
r = c_1 \frac{1 - s}{c_2 + s} \quad \text{with} \quad c_2 = 1 + \frac{2c_1}{r_{\text{max}}},
\]

where \( c_1 \) is a constant such that approximately half of the nodes are concentrated in the interval \( 0 \leq r \leq c_1 \). This transformation allows large values of \( r \) to be taken into account with relatively few basis functions. The domain is thus discretized in \( N \) points, \( N \) being the number of Chebyshev polynomials in which \( S \) has been expanded. With this discretization, (2.7)–(2.16) becomes an algebraic nonlinear eigenvalue problem which is solved using the linear companion matrix method described by Bridges & Morris (1984). The method consists in adding to (2.5) the eigenfunction \( aS \), obtaining thus a linear eigenvalue problem, but at the price of doubling the size of the original nonlinear one. The resulting (complex) linear eigenvalue problem is solved with double precision using an eigenvalue solver from the IMSL library (subroutine DGVCCG), which provides the entire eigenvalue and eigenvector spectrum. Spurious eigenvalues
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\[ N \quad a = \gamma + i\alpha \quad k = k_r + ik_i \text{ (OS)} \]

\begin{tabular}{|c|c|c|}
\hline
\( N \) & \( a \) & \( k \) \\
\hline
40 & 0.222 + 0.365i & 0.367202058 - 0.220458484i \\
60 & 0.2229717 + 0.365176287i & 0.367202508 - 0.220457875i \\
70 & 0.222971712 + 0.3651762873i & 0.367202981 - 0.220457371i \\
80 & 0.22297171238 + 0.365176287330i & 0.367203916 - 0.220456594i \\
100 & 0.2229717123848892 + 0.3651762873306434i & \\
120 & 0.2229717123848892 + 0.3651762873306434i & \\
\hline
\end{tabular}

Table 1. Convergence behaviour of \( a = \gamma + i\alpha \) for the most unstable mode with \( n = -1 \) and \( \omega = -0.15 \) for \( Re = 667, q = 0.7, W_0 = 0 \), as a function of the number \( N \) of Chebyshev polynomials. \( c_1 = 3, r_{max} = 100 \). Also shown are the results of Olendraru & Sellier (2002) for the same case (OS).

were discarded by comparing the computed spectra for an increasing number \( N \) of collocation points. A first selection of physical modes is made by discarding all the eigenvalues corresponding to eigenfunctions that do not die conveniently as \( r \to \infty \); that is, we consider only those eigenfunctions satisfying

\[
\frac{N/10}{\sum_{i=1}^{N} |F(r_i)|^2} < T, \quad (2.18)
\]

where the \( r_i \) are the radial nodes and \( T \) is a given tolerance.

2.3. Convergence tests

To check the efficiency and accuracy of the numerical method, we present convergence histories for two cases. The first one (table 1) is an inviscid mode already documented in the spatial stability analysis by Olendraru & Sellier (2002, see their table 1), corresponding to \( Re = 667 \) and \( q = 0.7 \), for the azimuthal wavenumber \( n = -1 \) and frequency \( \omega = -0.15 \), obtained with the same numerical parameters \( (c_1 = 3 \text{ and } r_{max} = 100) \) as those of Olendraru & Sellier (2002). It is observed that the eigenvalue is obtained with 10 significant digits for \( N = 100 \) collocation points (for \( \alpha \)), and with 12 significant digits for \( N = 120 \). In the notation of Olendraru & Sellier (2002), the complex wavenumber \( k \) is related to our complex eigenvalue \( a \) through \( a = ik \); i.e. \( -k_i = \gamma \) is the growth rate, and \( k_r = \alpha \) is the axial wavenumber. The convergence of the numerical results Olendraru & Sellier (2002) is less good than in our results (5 significant figures for \( N = 100 \)), and only the two first significant digits coincide with our results. This is probably due to the differences in the eigenvalue solvers and in the machine precisions used. In any case, the coincidence in two significant figures with Olendraru & Sellier (2002) and the excellent convergence history as \( N \) increases, make us confident of our numerical results.

For the viscous modes with large \( q \) (\( \geq 1.5 \)) and \( Re \) of interest here, more precision is required because the eigenfunctions are more involved. In addition, they are centre modes, concentrated near the vortex axis (see figure 1). For these reasons, we have used in the computations \( N = 140 \) and \( c_1 = 1 \), while maintaining \( r_{max} = 100 \). For very high Reynolds numbers \( (Re \geq 10^7) \), we have concentrated the nodes near the axis even more, using a factor \( c_1 = 0.1 \) (see figure 1b). For the highest Reynolds number considered, \( Re = 10^8 \), we have used \( N = 180 \), together with \( c_1 = 0.1 \) and \( r_{max} = 100 \). The second convergence history presented here (table 2) is for this last Reynolds number, \( Re = 10^8 \), with \( q = 3, \omega = -2.75, W_0 = 0 \) and \( n = -1 \). Note that
Figure 1. Real (Re) and imaginary (Im) parts of the eigenfunctions for (a) $Re = 10^4$, $q = 3$, $W_0 = 0$, $n = -1$, $\omega = -2.75$ ($N = 140, c_1 = 1$), and (b) $Re = 10^8$, $q = 3$, $W_0 = 0$, $n = -1$, $\omega = -2.75$ ($N = 180, c_1 = 0.1$).

Table 2. Convergence behaviour of $a = \gamma + i\alpha$ of the most unstable mode with $n = -1$ and $\omega = -2.75$ for $Re = 10^8$, $q = 3$, $W_0 = 0$, as a function of the number $N$ of Chebyshev polynomials. $c_1 = 0.1$, $r_{max} = 100$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$a = \gamma + i\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>160</td>
<td>$1.69 \times 10^{-3} + 0.246667i$</td>
</tr>
<tr>
<td>180</td>
<td>$1.69658 \times 10^{-3} + 0.246667i$</td>
</tr>
<tr>
<td>200</td>
<td>$1.696786 \times 10^{-3} + 0.24666715i$</td>
</tr>
</tbody>
</table>

Finally, to discard the spurious modes, a tolerance $T = 10^{-11}$ in (2.18) was used in all the reported cases.

3. Results

For real values of the frequency $\omega$ (spatial analysis), the governing stability equations have the symmetry property $a(\omega; n; Re, q) \mapsto a^*(-\omega; -n; Re, q)$, where the asterisk indicates the complex conjugate. Thus, if we allow for both positive and negative values of the forcing frequency $\omega$, only the cases with non-positive (or non-negative) azimuthal wavenumber $n$ have to be considered. It will be assumed that $n \leq 0$ in what follows. A spatial mode with $\omega < 0$, $n < 0$, and the eigenvalue $a = \gamma + i\alpha$ (for given values of $Re$ and $q$), physically corresponds to a spatial mode with the positive frequency $-\omega$, the positive azimuthal wavenumber $-n$, the axial wavenumber $-\alpha$, and the same spatial growth rate $\gamma$. On the other hand, only positive values of the swirl parameter $q$ will be considered, for the spatial stability equations are invariant under the transformation $(n, q) \mapsto (-n, -q)$, $(F, G, H, \Pi) \mapsto (F, -G, H, \Pi)$ (see, e.g. Olendraru & Sellier 2002). Thus, the stability properties of a vortex with swirl parameter $-q$ ($q > 0$), for azimuthal number $n$ ($n \leq 0$), and any frequency $\omega$ (positive or negative) are the same as those for a vortex with $q$, $-n$ and $-\omega$; i.e. the same as for $q$, $n$ and $-\omega$.

For $q \geq 1.5$ and given values of the parameters, we shall look for the least stable, or the most unstable, spatial modes propagating towards increasing $z$. That is, for...
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Figure 2. (a) $\gamma(\omega)$, and (b) $\alpha(\omega)$ for the less stable viscous modes for $q = 3$, $W_0 = 0$, $n = -1$, and different values of $Re$, as indicated.

each $q \geq 1.5$, $Re > 0$, $n \leq 0$, and a given positive, negative or zero value of $\omega$, we search for the largest value of $\gamma$ corresponding to a mode with a positive real part of the group velocity, which in its dimensionless form is given by

$$c_g \equiv \frac{\partial \omega}{\partial \alpha}. \quad (3.1)$$

If $\gamma < 0$ ($\gamma > 0$), the amplitude of the wave packet corresponding to the selected forcing frequency $\omega$, which moves downstream at the real group velocity $c_g > 0$, will decrease (increase) with $z$, and the flow will be spatially stable (unstable). Thus, a spatial growth rate $\gamma > 0$ with $c_g > 0$ corresponds to a convectively unstable flow, since the growing perturbation is advected downstream of the source with the forcing frequency $\omega$, leaving the flow in its undisturbed state when the forcing ceases (see, e.g. Huerre & Monkewitz 1990).

Here, we characterize the viscous unstable modes previously found by Fabre & Jacquin (2004) for $W_0 = 0$ (for large $q$ and large $Re$), but using the present spatial stability formulation instead of a temporal one, and for different values of $W_0$, which is now a relevant parameter in the spatial stability analysis. In particular, we shall consider only swirl numbers $q \geq 1.5$, so that all unstable modes (if any) are necessarily viscous modes (Lessen et al. 1974). In fact, we find that at most only one viscous mode may become unstable in all the cases considered for $q > 1.5$.

3.1. Viscous modes for $W_0 = 0$ (convective instabilities)

We start with the case $W_0 = 0$. Thus, for instance, figure 2 shows the dispersion relations, $\gamma(\omega)$ and $\alpha(\omega)$, for the unstable viscous modes with azimuthal wavenumber $n = -1$, corresponding to $q = 3$ and different Reynolds numbers. They become unstable at $Re_c \approx 8200$ for $\omega_c \approx -2.75$, and remain unstable for $R > Re_c$, though they become neutrally stable as $Re \to \infty$ because they are viscous modes (note that the growth rate $\gamma$ first increases with $Re$, reaches a maximum at about $Re \approx 10^5$, and then decreases very slowly as $Re \to \infty$). The functions $\alpha(\omega)$ are always almost linear functions, so that the phase speeds $c = \omega/\alpha$ practically coincide with their group
Figure 3. Regions of instability in the $(\omega, q)$-plane for the most unstable mode with $n = -1$, $n = -2$ and $n = -3$ for different values of $Re$: (a) $10^3$, (b) $10^4$, (c) $10^5$, (d) $10^6$, (e) $10^7$, (f) $10^8$. The outermost continuous lines correspond to $\gamma = 0$ (neutral curves), within which we have plotted contour lines for $\gamma > 0$ (the increment in $\gamma$ is 0.005 in all the cases plotted). The dashed lines correspond to the asymptotic approximation for large $Re$ of the upper and lower neutral curves given by Le Dizès & Fabre (2007).

speeds $c_g = \partial \omega / \partial \alpha$, being both obviously positive (we look only for eigenvalues with $c_g \geq 0$). These features are common for all the values of $q \geq 1.5$ (viscous modes). Thus, no absolutely unstable modes are found, for $c_g$ never vanishes for these viscous modes in all the cases considered here with $W_0 = 0$.

The instability regions in the $(\omega, q)$-plane for the viscous modes with $n = -1$, $-2$ and $-3$ are summarized in figure 3 for different (high) values of the Reynolds number. For the present case with $W_0 = 0$, these viscous modes with $q > 1.5$ were not
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reported in the spatial stability analysis of Olendraru & Sellier (2002). They are the spatial counterparts of the temporal viscous modes found by Fabre & Jacquin (2004), though these authors mapped the unstable region up to $Re \approx 10^6$, and we report here results up to $Re = 10^8$. For each Reynolds number and $n$, there exists a critical swirl number, $q_c(Re)$, above which the flow is stable. For instance, for $Re = 10^4$, we find that $q_c = 3.233$, that practically coincides with the value $3.235$ reported by Fabre & Jacquin (2004). This critical swirl number is plotted as a function of the Reynolds number in figure 4 for $n = -1, -2$ and $-3$.

It is worth noting in figure 3 that the instability regions narrow as the Reynolds number increases, contracting to a very slender region around the straight line $\omega = nq$ for $Re \to \infty$, in accordance with the lowest order of the asymptotic results for large Reynolds numbers given by Le Dizès & Fabre (2007). These asymptotic results were a useful initial guide to finding out the frequency regions in which the present viscous modes were unstable for each $q$. We have include in figure 3 the asymptotic approximation to the upper and lower branches of the neutral curves. For the present $q$-vortex (1.1), and up to the order of $Re^{-1/2}$, the upper and lower parts of the neutral curve (for $q \geq 1.5$ and $n \leq 0$) are given by (Le Dizès & Fabre 2007)

$$\omega \sim nq + \alpha (1 + W_0) - \frac{Re^{-1/2}}{2} \sqrt{3n(2/q - q)}, \quad (3.2)$$

with

$$\alpha \sim -n/q, \quad (3.3)$$

for the upper branch of the neutral curve, and

$$\alpha \sim 0, \quad (3.4)$$

for the lower branch. The dashed lines in figure 3 are given by (3.2) with $W_0 = 0$. Though these asymptotic expressions are obtained by Le Dizès & Fabre (2007) from a temporal stability analysis, the present neutral curve of the spatial stability analysis is easily found by setting to zero the imaginary part of the frequency (the growth rate),

![Figure 4. Critical swirl number above which no unstable viscous modes exist as a function of $Re$ for several values of $n$. The thin continuous lines correspond to the asymptotic results (Fabre & Le Dizès 2007): $q_c/Re^{1/3} \approx 0.1408$ for $n = -1$, 0.1142 for $n = -2$ and 0.0858 for $n = -3$.](image)
yielding the axial wavenumber (3.3) or (3.4), and the corresponding real frequency (3.2). The agreement between asymptotic and numerical results obviously improves as \( Re \) increases. Thus, it is very good for \( Re \geq 10^5 \), approximately, but it is poor for \( Re = 10^3 \), especially for the lowest branch of the neutral curve.

As mentioned above, figure 4 depicts the critical (maximum) swirl numbers as functions of the Reynolds number. These values practically coincide with those plotted in figure 3 of Fabre & Jacquin (2004) (up to \( Re = 10^5 \)). The corresponding frequencies are plotted in figure 5. For each \( Re \), the largest critical swirl number corresponds always to the helical mode with \( n = -1 \). It is observed that both \( q_c \) and \( |\omega_c| \) scale as \( Re^{1/3} \) for large \( Re \), as predicted by the asymptotic analysis. We have included in figure 4 these asymptotic results for \( q_c \) (Fabre & Le Dizès 2007).

Finally, to close this section, it is important to mention that no unstable axisymmetric \((n = 0)\) viscous modes were found for \( q \geq 1.5 \).

3.2. Viscous modes and the onset of absolute instabilities for \( W_0 \neq 0 \)

We consider now the viscous unstable modes when an uniform axial flow is present in the vortex, \( W_0 \neq 0 \). Particularly, we pay special attention to wake-like vortices with \( W_0 < 0 \), for which, as we shall see, absolute instabilities may be present in the flow (as previously found by Delbende, Chomaz & Huerre 1998 for inviscid modes with \( q < 1.5 \), and hinted for viscous modes for \( Re = 10^4 \) by Olendraru & Sellier 2002). We also consider the cases with \( W_0 > 0 \), but they never become absolutely unstable.

Figure 6 shows the instability regions in the \((\omega, q)\)-plane for the viscous modes \((q > 1.5)\) with \( n = -1 \), \( Re = 10^3 \), and different values of \( W_0 \) ranging between \(-1.25\) and \(+0.25\). It is observed that the critical (maximum) swirl number coincides for all the values of \( W_0 \), \( q_c^{n=-1}(Re = 10^3) \approx 1.788 \). This is because the temporal stability analysis, contrary to the present spatial analysis, does not depend on \( W_0 \), so that the stability boundary in the parameter plane \((q, Re)\) (depicted in figure 4) is common for all the values of \( W_0 \), for given values of \( n \).

Inside the instability regions depicted in figure 6, flows with \( W_0 \neq 0 \) may become absolutely unstable below some critical swirl number. To find these absolute instabilities, we look for saddle points in the dispersion relation, with a cusp point in the spatial growth rate \( \gamma(\omega) > 0 \) (e.g. Fernandez-Feria & del Pino 2002), and apply the Briggs–Bers criterion (see, e.g. Huerre & Monkewitz 1990). To that end, we look
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Figure 6. Regions of instability (neutral curves, $\gamma = 0$) in the $(\omega, q)$-plane for the most unstable mode with $n = -1$ for $Re = 10^3$, and for different values of $W_0$, as indicated.

Figure 7. Saddle point in the dispersion relation for the case $Re = 10^3$, $n = -1$, $W_0 = -1$, which corresponds to an absolute instability for $q = q_{ca} \simeq 1.775$ and $\omega = \omega_0 \simeq -1.6151$.

for the dispersion relation functions [$\gamma(\omega), \alpha(\omega)$] for diminishing values of $q$, starting from $q = q_c$. For the cases depicted in figure 6, we have found absolute instabilities for $W_0 = -0.75$ and for $W_0 = -1.0$. For instance, for $W_0 = -1$, the saddle point in the dispersion relation is shown in figure 7, which corresponds to an absolute critical swirl $q_{ca} \simeq 1.775$, and an absolute frequency $\omega_0 \simeq -1.6151$. These critical values are marked with an asterisk inside the convective instability regions in figure 8 for $W_0 = -0.75$ and $W_0 = -1$. For $q_{ca} < q < q_c$, the flow is just convectively unstable (in the frequency range inside the curves depicted in figure 8), whereas for $q < q_{ca}$, the flow may be absolutely unstable in a frequency range which obviously lies inside the convectively unstable range. We have plotted in figure 8 only the neutral curve for convective instability and the line with the maximum growth rate until it ends at
the asterisk marking the onset of absolute instability, corresponding to $q_{ca}$ and the absolute frequency $\omega_0$.

The analysis has been carried out for other values of $Re$ and $n$. For instance, figure 9 shows the convective instability regions, and the values of $q_{ca}$ and $\omega_0$, for $n = -1$, $Re = 10^4$ and a couple of values of $W_0$. It is seen that $q_{ca}$ approaches $qc$ as the Reynolds number increases for $W_0 = -1$. All the values of $q_{ca}$ and $\omega_0$ are plotted in figure 10 as functions of $W_0$ for $n = -1$ and several values of $Re$ up to $10^6$. Figure 10(a) marks, in fact, the absolutely unstable regions in the plane $(q, W_0)$ for several values of $Re$. It is observed that these regions increase appreciably in size, reaching higher values of the swirl number, as $Re$ increases. For $Re = 10^4$ practically coincides with the region depicted in figure 24 of Olendraru & Sellier (2002), with a maximum value of the swirl number $q_{ca,max}(Re = 10^4, n = -1) \approx 3.23$. This maximum value of the critical swirl for absolute instability is always reached for $W_0 \simeq -1$, independently of $Re$ ($W_0 = -1$ marks the boundary between co-flowing and counter-flowing wakes). Thus, Batchelor’s vortex wakes with $W_0 = -1$ are the globally most unstable members of the family of vortices. This is easy to understand since, in

Figure 8. Regions of instability (neutral curves, $\gamma = 0$) in the $(\omega, q)$-plane for the most unstable mode with $n = -1$ for $Re = 10^3$, and (a) $W_0 = -0.75$ and (b) $-1$. The asterisks inside the curves mark the frequency ($\omega_0$) and swirl parameter ($q_{ca}$) for the onset of absolute instability (the lines ending at the asterisks correspond to the maximum growth rate for convective instabilities).

Figure 9. As in figure 8, but for $Re = 10^4$. 
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Figure 10. (a) $q_{ca}$, and (b) $\omega_0$ as functions of $W_0$ for $n = -1$ and several values of $Re$ (as indicated).

Figure 11. As in figure 10, but for $n = -2$.

this case, the vortex centre is a stagnant region, and the instabilities considered here are localized in this region.

Figures 11 and 12 show the critical values for absolute instabilities for modes with azimuthal wavenumbers $n = -2$ and $n = -3$ as functions of $W_0$. The maximum values of $q_{ca}$ for each $Re$ are also reached at $W_0 \simeq -1$, as in the $n = -1$ case, except for the lower Reynolds number considered ($Re = 10^3$) in the case $n = -3$ (see figure 12). All the maximum values of $q_{ca}$ as functions of $Re$ are plotted in figure 13 for $n = -1, -2$ and $-3$. As in the case of convective instabilities, helical waves with $n = -1$ are those with the largest region of absolute instability for Batchelor’s vortices. It is observed that, for large $Re$, $q_{ca,max}$ scales as $Re^{1/3}$. In fact, as seen in figure 8 for $Re = 10^4$, $q_c$ and $q_{ca}$ almost coincide for large $Re$ when $W_0 = -1$, so that this asymptote coincides with that shown in figure 4, and with the asymptotic results of Fabre & Le Dizès...
Figure 12. As in figure 10, but for \( n = -3 \).

Figure 13. \( q_{ca,\text{max}} \) as a function of \( Re \) for the three values of \( n \) considered. For large \( Re \), these curves behave as \( q_{ca,\text{max}} \approx c_n Re^{1/3} \) (dashed lines), with the constants \( c_n \) for each \( -n \) approximately equal to those given in the caption of figure 4.

(2007). This could have been anticipated because whenever the flow is unstable, there is one reference frame, which in this problem corresponds to one value of \( W_0 \) (= -1 in this case), in which the instability is absolute. Finally, figure 14 shows \( q_{ca} \) as a function of \( Re \) for the three values of \( n \) considered and for \( W_0 = -1.125 \), and -0.75 (the case \( W_0 = -1 \) is not included because it coincides with figure 13, except for the point corresponding to \( Re = 10^5 \) for \( n = -3 \)). In the case \( W_0 = -1.125 \), no curve for \( n = -3 \) is given because the flow is not absolutely unstable for this \( W_0 \) for any value of \( Re \) (see figure 12). For large \( Re \), \( q_{ca} \) scales as \( Re^{1/3} \), similarly to the case \( W_0 = -1 \), with proportionality constants given in the figure caption. This type of asymptotic behaviour for large \( Re \) is also found for the case \( W_0 = -0.75 \), but only for \( n = -1 \). For \( n = -2 \) and \( n = -3 \), the values of \( q_{ca} \) are now low and so close to the viscous
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Figure 14. $q_{ca}$ as a function of $Re$ for $n = -1$, $-2$ and $-3$, and for (a) $W_0 = -0.75$ and (b) $W_0 = -1.125$. With dashed lines, we show the asymptotic behaviours for large $Re$, $q_{ca} \approx c_n Re^{1/3}$, if that is the case: $c_1 \approx 0.0965$ for $W_0 = -0.75$; $c_1 \approx 0.1333$ and $c_2 \approx 0.0906$ for $W_0 = -1.125$.

instability boundary $q \approx 1.5$ that the asymptotic behaviour for the viscous centre modes discussed by Le Dizès & Fabre (2007) is no longer valid.

4. Summary and conclusions

We have characterized the viscous instabilities of Batchelor’s vortex for high swirl numbers. In particular, we have considered $q > 1.5$, a range of the swirl parameter in which Batchelor’s vortex does not present inertial instabilities (Mayer & Powell 1992). These viscous instabilities, which are centre modes concentrated near the axis of the vortex, were found by Fabre & Jacquin (2004) using a temporal stability analysis. Here, we use a spatial stability analysis which, although it is numerically more involved, since one has to tackle a nonlinear eigenvalue problem, has the advantage of directly providing relevant physical information such as the frequency ranges of the unstable modes in terms of the other parameters of the flow, namely the Reynolds
number and the swirl parameter, and in terms of the azimuthal wavenumber of the perturbations. It also provides the onset of absolute instability in a straightforward way.

We have fully characterized the convective viscous instabilities from $Re = 10^3$, which is approximately the lowest Reynolds number for which the viscous instabilities for large $q (> 1.5)$ do appear, up to the remarkably high value $Re = 10^8$. For these very high Reynolds numbers we find excellent agreement with the asymptotic analysis of Le Dizès & Fabre (2007). We find that the most unstable modes are helical ones with azimuthal wavenumber $n = -1$. The critical, or maximum, swirl number for convective instability scales as $Re^{1/3}$ for large $Re$, as predicted by the asymptotic analysis (Le Dizès & Fabre 2007). No axisymmetric ($n = 0$) unstable viscous modes are found.

We also characterize the absolute/convective instability boundary for these viscous modes. Again, helical modes with $n = -1$ are the first to become absolutely unstable as $q$ decreases, for each $Re$. Absolute instabilities are present only when the uniform axial velocity superimposed to the vortex, $W_0$, is negative, i.e. for wakes. In particular, the case $W_0 = -1$, which corresponds to the axial velocity that marks the boundary between co-flowing and counter-flowing wakes, is always the most absolutely unstable one for every $Re$ and $n$, as could have been anticipated because, in this case, the vortex centre, where the present instabilities are localized, is a stagnant region. For large $Re$, the maximum swirl number for absolute instability (occurring for $W_0 = -1$) also scales as $Re^{1/3}$. Actually, we show that, for this case $W_0 = -1$, the critical swirls for convective and absolute instabilities practically coincide.

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Non-parallel spatial stability of Batchelor vortex

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Abstract

We analyze the spatial stability of the so-called Batchelor’s vortex taking into account non-parallel effects associated to the axial variation of this self-similar vortex that models the far downstream behavior of trailing vortices. To that end we integrate the Parabolized Stability Equations (PSE) along the axis of the vortex starting from the local ”near-parallel” stability results (eigenvalues and eigenfunctions) at a given axial location. We first discuss the differences with the parallel version of Batchelor’s vortex, also called q-vortex, which is the standard local version of this vortex used in previous stability analysis. We then characterize the non-parallel stability properties of Batchelor’s vortex along its axis for several cases of interest in trailing vortices, particularly for actual conditions in the far wake behind large commercial aircrafts. As the main result, we obtain that the PSE method predicts a much shorter stabilization axial distance of the viscous unstable modes present at these conditions than the axial distance predicted by near-parallel stability analyses.
I. INTRODUCTION

The stability properties of aircraft trailing wakes are crucial to predict their far-field behavior and, therefore, to evaluate their potential impact on a following aircraft.\textsuperscript{1,2} Batchelor\textsuperscript{3} derived a similarity solution for the flow in a trailing vortex far downstream that has been traditionally used as a simple model for the far field of trailing vortices with axial flow. However, a parallel-flow simplification of Batchelor’s vortex, usually called $q$-vortex, has been commonly used for stability analyses,\textsuperscript{4–13} thus failing to account for non-parallel effects which may be very important in the far-field behavior of these vortices. In the present work we present a non-parallel spatial stability analysis of Batchelor’s vortex that will fill this gap in the knowledge on the stability and far-field behavior of these trailing vortices.

II. FORMULATION OF THE PROBLEM AND NUMERICAL METHOD

We use non-dimensional cylindrical coordinates $(r, \theta, z)$, where a characteristic radius $r_c$ and a characteristic axial length $z_c$ are used to render dimensionless $r$ and $z$, respectively (see below). We assume that the aspect ratio

$$\Delta \equiv \frac{r_c}{z_c}$$

(1)

is a small parameter. The velocity field $(U, V, W)$ in the self-similar Batchelor’s solution is made dimensionless with the uniform axial velocity far away from the axis of the vortex, $W_\infty$, and can be written as\textsuperscript{3}

$$U(r, z) = 0, \quad V(r, z) = \frac{q_1}{r} \left(1 - e^{-r^2/z}\right),$$

(2)

$$W(r, z) = 1 + \frac{q_1^2}{2z} Q(\frac{r^2}{z}) - \left[q_1^2 \ln \left(\frac{Re_1^2 z}{4}\right) + \frac{Re_1}{4\delta}\right] \frac{e^{-r^2/z}}{2z},$$

(3)

where

$$Q(\eta) \equiv e^{-\eta}[\ln \eta + E_1(\eta) - 0.807] + 2E_1(\eta) - 2E_1(2\eta),$$

(4)

being $E_1(\eta) \equiv \int_{\eta}^{\infty} dx \, e^{-x} / x$ the exponential integral function.\textsuperscript{14}

The viscous core radius along Batchelor’s vortex is proportional to $\sqrt{4\nu z/W_\infty}$, where $\hat{z}$ represents the dimensional axial length. Therefore, we have used

$$r_c \equiv \sqrt{\frac{4\nu z_c}{W_\infty}}$$

(5)

as the characteristic radius. Consequently, the Reynolds number and the swirl parameters are

$$Re_1 = \frac{W_\infty r_c}{\nu}, \quad q_1 = \frac{\Gamma_0}{2\pi r_c W_\infty} \equiv \frac{V_c}{W_\infty},$$

(6)

where $\Gamma_0$ is the circulation of the vortex for $r \to \infty$. Subscript 1 has been used in (6) to distinguish these non-dimensional parameters from those commonly used in the parallel
q-vortex (see below). Due to the choice (5) for \( r_c \), the aspect ratio (1) is not an independent parameter, but it is related to the Reynolds number through

\[
\Delta = \frac{4}{Re_1},
\]

so that it will not appear explicitly in the present formulation. Finally, the non-dimensional parameter \( \delta \) in (3) is related to the constant \( L \) with dimensions of area defined in Ref.\(^3\) by

\[
\delta \equiv \frac{z e^{r_c}}{L}.
\]

Batchelor’s vortex (2)-(3) may be compared to its parallel version, usually called q-vortex,

\[
U = 0, \quad V = \frac{q}{r} \left( 1 - e^{-r^2} \right), \quad W = - \left( W_0 + e^{-r^2} \right).
\]

The local swirl parameter \( q \), and the corresponding local Reynolds number \( Re \), are now functions of the axial coordinate because the velocity field (9) is made dimensionless with a characteristic axial velocity given by the difference between \( W_\infty \) and the axial velocity at the axis, which depends on \( z \). One has the following relations between these non-dimensional parameters:

\[
q = \frac{q_1}{F(z)}, \quad Re = Re_1 F(z),
\]

where \( F(z) \) is the ratio between the two different characteristic axial velocities used in (2)-(3) and in (9),

\[
F(z) \equiv \frac{q_1^2}{2} \ln \left( \frac{Re_1^2 z}{4} \right) + \frac{Re_1}{8\delta}.
\]

The local, non-dimensional external axial velocity \( W_0 \) in (9) is given, approximately (neglecting the weak radial variation), by

\[
W_0 \simeq -\frac{z}{F(z)}.
\]

With these equivalences, the azimuthal velocity profiles in (2) and (9) coincide at \( z = 1 \), while the differences in the axial velocity profiles depend on \( q_1 \): For large \( q_1 \), the second term in the r.h.s. of Eq. (3) may become important, modifying the axial velocity profile in relation to the q-vortex (9) and, therefore, its stability properties. We shall see, however, that \( q_1 \) is usually small for actual trailing vortices of interest, though \( q \) is normally order unity or larger.\(^2,9,10\)

For generality sake we shall formulate the spatial stability problem for a general vortex with velocity field \([\Delta U(r, z), V(r, z), W(r, z)]\), where the radial velocity is not zero like in (2), but \( O(\Delta) \) (it has been re-scaled with \( \Delta \), so that \( U \) is, according to the continuity equation, of order unity, like \( W \)). We shall make use of the fact that \( \Delta \ll 1 \) to simplify the stability equations, neglecting second order derivatives in the axial coordinate, an approximation that constitutes the basis of the so-called Parabolized Stability Equations (PSE for short).\(^{15,16}\) One can write the non-dimensional velocity and pressure fields as follows:
\[
\begin{pmatrix}
u(r, \theta, z, t) \\
v(r, \theta, z, t) \\
w(r, \theta, z, t) \\
p(r, \theta, z, t)
\end{pmatrix} = \begin{pmatrix}
\Delta U(r, z) \\
V(r, z) \\
W(r, z) \\
P(r, z)
\end{pmatrix} + \begin{pmatrix}
u'(r, \theta, z, t) \\
v'(r, \theta, z, t) \\
w'(r, \theta, z, t) \\
p'(r, \theta, z, t)
\end{pmatrix},
\tag{13}
\]

where capital letters are used for the basic flow and primed variables for the perturbations. The pressure \( P \) of the basic flow has not been written down because it does not enter explicitly into the linear stability equations. The perturbations \( s \equiv [u', v', w', p']^T \) are decomposed in the standard form

\[
s(r, \theta, z, t) = S(r, z) \chi(z, \theta, t),
\tag{14}
\]

where the complex amplitude

\[
S(r, z) \equiv \begin{pmatrix}
F(r, z) \\
G(r, z) \\
H(r, z) \\
\Pi(r, z)
\end{pmatrix}
\tag{15}
\]
is allowed to depend on the axial co-ordinate \( z \), in addition to the radial one, to account for the non-parallelism of the basic flow. The other part of the perturbation is an exponential that describes the wave-like nature of the disturbances,

\[
\chi(z, \theta, t) = \exp \left[ \frac{1}{\Delta} \int_{z_0}^{z} a(z')dz' + in\theta - i\omega t \right],
\tag{16}
\]

where \( z_0 \) is the axial point in which the disturbances are introduced, \( a(z) \) is the nondimensional (complex) axial wavenumber, \( n \) is the azimuthal wave number, and \( \omega \) is the nondimensional frequency of the disturbances. One may use \( z_0 \) as the initial axial length, so that \( z_0 = 1 \) in (16). \( a \) and \( \omega \) are defined as

\[
a \equiv ir_c \hat{k} \equiv \gamma + i\alpha, \quad \omega \equiv \frac{\hat{\omega}r_c}{W_\infty},
\tag{17}
\]

where \( \hat{k} \) and \( \hat{\omega} \) are the dimensional axial wavenumber and frequency, respectively. The real part of \( a(z) \), \( \gamma(z) \), is the exponential growth rate, and its imaginary part, \( \alpha(z) \), is the axial wavenumber. In the spatial stability analysis to be considered here, one fixes a real frequency \( \omega \) and looks for complex values of \( a(z) \). The flow is (convectively) unstable when \( \gamma(z) > 0 \). Finally, the azimuthal wave number \( n \) is equal to zero for axisymmetric perturbations, and different from zero for non-axisymmetric perturbations.

Substituting (14)-(15) into the incompressible Navier-Stokes equations, and neglecting second-order terms in both the small perturbations (i.e., linear stability), and \( \Delta \) (i.e., neglecting terms with second order axial derivatives, which constitutes the basis of the PSE technique) one obtains the following parabolic stability equation for \( S \) (more precisely, we neglect terms of the order of \( \chi^2, \Delta^2, \) and \( \Delta/Re, \) or smaller):

}\]
\[ L \cdot S + \Delta M \cdot \frac{\partial S}{\partial z} = 0, \]  
\[ L \equiv L_1 + aL_2 + \frac{1}{Re} L_{31} + \Delta L_{32} + a^2 \frac{1}{Re} L_4, \]  
\[ L_1 = \begin{pmatrix} \frac{1}{r} + \frac{\partial}{\partial r} & \frac{iv}{r} & 0 & 0 \\ i \left( \frac{iv}{r} - \omega \right) & -\frac{2V}{r} & 0 & \frac{\partial}{\partial r} \\ \frac{\partial V}{\partial r} + \frac{V}{r} & i \left( \frac{iv}{r} - \omega \right) & 0 & \frac{iv}{r} \\ \frac{\partial W}{\partial r} & 0 & i \left( \frac{iv}{r} - \omega \right) & 0 \end{pmatrix}, \]  
\[ L_2 = M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]  
\[ L_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -D_r^2 + \frac{n^2+1}{r^2} & \frac{2in}{r^2} & 0 & 0 \\ -\frac{2in}{r^2} & -D_r^2 + \frac{n^2+1}{r^2} & 0 & 0 \\ 0 & 0 & 0 & -D_r^2 + \frac{n^2}{r^2} \end{pmatrix}, \quad D_r^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}, \]  
\[ L_{32} = \begin{pmatrix} 0 & \partial U/\partial r & 0 & 0 \\ U \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) & \frac{\partial V}{\partial r} & 0 & 0 \\ 0 & 0 & \left( \frac{\partial W}{\partial r} + U \frac{\partial}{\partial r} \right) & 0 \end{pmatrix}. \]  

This equation has to be solved with the radial boundary conditions

\[ r \to \infty, \quad F = G = H = 0; \]  
\[ r = 0, \quad F = G = 0, \quad dH/dr = 0, \quad (n = 0), \]  
\[ F = \pm iG = 0, \quad dF/dr = 0, \quad H = 0, \quad (n = \pm 1), \]  
\[ F = G = H = 0, \quad (|n| > 1).^{17} \]

It also needs an initial condition at \( z = z_0 \), which may be taken unity without loss of generality. A convenient choice is the solution of the local eigenvalue problem\(^{15,18}\)

\[ L_0 \cdot S_0 \equiv \left[ L_1 + a_0 L_2 + \frac{1}{Re} L_{31} + \Delta L_{32} + a_0^2 \frac{1}{Re} L_4 \right] \cdot S_0 = 0, \]
that provides the initial eigenvalue $a_0 \equiv a(z_0)$, and eigenfunction $S_0(r) \equiv S(r, z_0)$, which will be used to start the axial integration of Eq. (18) for a given set of nondimensional parameters. Equation (28) accounts for the effect of the nonparallelism of the basic flow, but neglects the effect of the history or convective evolution of the perturbations. Its solution for different values of $z > z_0$ will be compared with the solution to the PSE (18) to measure the relative importance of this last effect. This local solution will be termed near-parallel (NP) solution. It should not be confused with the full parallel (P) solution, obtained from (28) when one neglects the term proportional to $\Delta$, that therefore does not take into account even the nonparallelism of the basic flow. The NP solution will also be used as a reference to the PSE analysis. Finally, it is worth to mention that the spatial stability analysis considered here is the only appropriate one for a stability analysis accounting for non-parallel effects based on the PSE.$^{15}$

As it stands, there is some ambiguity in the partition of the perturbations (14) into two functions of $z$. To close the problem one has to enforce an additional condition which puts some restriction on the axial variation of $S$. Basically, one uses a normalization condition that restricts rapid changes in $z$ of $S$, according to the slow axial variation of the basic flow (small $\Delta$). Thus, the growth rate and the axial sinusoidal variation are represented by the exponential function $\chi$. Several types of normalization conditions can be used.$^{15,16,18,19}$ Here we will use an integral condition based on the kinetic energy of the perturbations. Defining a physical amplification rate $a_1$ based on the axial variation of the kinetic energy of the perturbations,

$$a_1(z) \equiv \gamma_1(z) + i\alpha_1(z) = \Delta \frac{\int_0^\infty \left[ u'^\dagger \frac{\partial u'}{\partial z} + v'^\dagger \frac{\partial v'}{\partial z} + w'^\dagger \frac{\partial w'}{\partial z} \right] dr}{\int_0^\infty \left[ |u'|^2 + |v'|^2 + |w'|^2 \right] dr}$$

$$= a(z) + \Delta \frac{\int_0^\infty \left[ F'^\dagger \frac{\partial F'}{\partial z} + G'^\dagger \frac{\partial G'}{\partial z} + H'^\dagger \frac{\partial H'}{\partial z} \right] dr}{\int_0^\infty \left[ |F|^2 + |G|^2 + |H|^2 \right] dr}, \quad (29)$$

where $\dagger$ denotes the complex conjugate, the normalization condition used here can be expressed as $a_1(z) = a(z)$ for all $z > z_0$. That is, at each axial step in the integration of (18), the second term in the right hand side of (29) (the one multiplied by $\Delta$) is set equal to zero, transferring the main part of the streamwise variation of the perturbations to the exponential function $\chi$. For the numerical method used to solve the problem (18)-(29) we refer the reader to Refs.$^{18,19}$, and to Ref.$^{10}$ for the particular numerical details of the local eigenvalue problem (28) in the q-vortex.

### III. RESULTS

Our main aim in this work is to see how the non-parallelism of the basic flow affects the stability of trailing vortices in actual aircraft conditions. For this reason we consider a value of interest of the Reynolds number and characterize the stability properties for all the relevant swirl numbers and frequencies of the perturbations. In an aircraft trailing vortex, the relevant Reynolds number is based on the wing span $b$, i.e., $Re_b = W_\infty b / \nu$. Taking into account that Batchelor’s vortex is valid far from the wing tip, when the characteristic radius $r_c$ of the vortex is sufficiently large, which can be expressed as a given fraction $\epsilon$ of the wing span $b$, $r_c = \epsilon b^{20}$ we have from (6) that $Re_1 = \epsilon Re_b$. Typically, $\epsilon \ll 1,^{20}$ and
Re_b may range between $10^7$ and $10^9$ in commercial aircrafts. Therefore, we shall select the value $Re_1 = 10^5$, which is appropriate for the lower aircraft speeds in landing and taking off where the present stability analysis of trailing vortices is more relevant. For this Reynolds number we shall vary the swirl parameter $q_1$ and look for the most unstable modes with azimuthal wave numbers $n = -1, -2$, and $-3$ (note that the mode $n = -1$ is the first to become unstable for these high Reynolds numbers in the parallel flow approximation).\textsuperscript{9,10} The remaining non-dimensional parameter $\delta$ in (3) can be estimated by using the following approximate relation between the drag $D$ associated to the trailing vortex and $L$ (we neglect the term proportional to the square of the circulation):\textsuperscript{3}

\[
\frac{D}{\rho} \approx \frac{1}{2} \pi LW_s^2. \tag{30}
\]

Since $D = C_D \rho W_s^2 S / 2$, where $C_D$ is the drag coefficient and $S$ a wing reference area, using for the drag coefficient the relation $C_D \approx C_L^2 / (\pi A_R e)$, where $C_L$ is the lift coefficient ($\text{Lift} = C_L \rho W_s^2 S / 2$), $A_R = b^2 / S$ is the wing aspect ratio, and $e = 1$ for an elliptically loaded wing,\textsuperscript{21} one is left with

\[
L \approx \left( \frac{C_L b}{\pi A_R} \right)^2. \tag{31}
\]

The aspect ratio typically ranges between $A_R = 7$ and $A_R = 10$ for commercial aircrafts, while the lift coefficient $C_L$ ranges from 0.7 to 2.0.\textsuperscript{22} Substituting this expression, together with $r_c = \epsilon b$ and (5) into (8), one gets the following estimation for $\delta$:

\[
\delta \approx \left( \frac{\pi A_R}{2 C_L} \right)^2 \epsilon^3 Re_b = \left( \frac{\pi A_R}{2 C_L} \right)^2 \epsilon^2 Re_1. \tag{32}
\]

On the other hand, $q_1$ may also be related to $C_L$ and $A_R$ taking into account that the lift is equal to the flux of the wake vertical momentum, so that the circulation can be approximated by\textsuperscript{22}

\[
\Gamma_0 \approx \frac{C_L W_s b}{2 s_o A_R}, \tag{33}
\]

where $s_o$ is a coefficient relating the wing span with the separation $\hat{b}$ between the two trailing vortices, $\hat{b} = s_o b$. We shall use the value $s_o = \pi / 4$ for an elliptically loaded wing.\textsuperscript{21} The swirl parameter $q_1$ is then related to $\epsilon, C_L$ and $A_R$ through

\[
q_1 \approx \frac{C_L}{\pi^2 \epsilon A_R}. \tag{34}
\]

Summing up, we shall consider the stability properties of Batchelor’s vortex (2)-(3) for $Re_1 = 10^5$ and the azimuthal wave numbers $n = -1, -2$, and $-3$ as the swirl parameter $q_1$ is varied. The different values of $q_1$ are selected by varying $\epsilon$ in (34) with a typical value of the aspect ratio, $A_R = 10$, and the lift coefficient, $C_L = 0.7$, which also fix the remaining dimensionless parameter $\delta$ in (3) through (32). We are interested here in the \textit{viscous} modes for high Reynolds numbers and high swirl numbers previously found for the q-vortex.\textsuperscript{9–11,13} To have a first estimation of the critical swirl numbers, we use the large-Reynolds-number asymptotic approximation for the neutral curve in the $(\omega, q_1)$-plane near $q_{1c}$ given by Fabre
and Le Dizes\cite{LeDizes} for the viscous centre-modes in the parallel flow approximation. For Batchelor’s vortex at \( z = z_0 = 1 \), this neutral curve can be written as\cite{LeDizes}

\[
\omega \sim nq_1 + \frac{n(1 - F)}{2q_1} \left[ 0.192q_1^2 + F \pm \sqrt{(0.192q_1^2 + F)^2 + \frac{4k_n}{n} Re_1^{-1/2} q_1^{3/2}} \right],
\]

where \( F = F(z = 1) \) in \( (11) \), and \( k_n \approx 4.73, 12.96, \) and \( 29.86 \) for \( n = -1, -2, \) and \( -3 \), respectively. This equation also yields an implicit asymptotic approximation for the critical swirl,

\[
q_{1c}(Re_1) \sim \left( -n \frac{0.192q_{1c}^2 + F}{4k_n} \right)^{2/3} Re_1^{1/3}.
\]

Figure 1 shows with circles \( \omega(q_1) \) given by \( (35) \) for \( Re_1 = 10^5 \) and \( n = -1, -2, -3 \), together with the values computed numerically by solving the local eigenvalue problem \( (28) \) (lines). The mode with \( n = -1 \) is the first to become unstable as \( q_1 \) is increased, with \( q_{1c}^{-1} \approx 0.031 \) from \( (28) \); then the mode with \( n = -2 \) becomes unstable \( [q_{1c}^{-2} \approx 0.033] \), and, finally, \( n = -3 \ [q_{1c}^{-3} \approx 0.036] \). For \( n = -1 \) [Fig. 1(a)], the critical swirl computed numerically with the near-parallel approximation \( (28) \) is very close to the asymptotic results \( (36) \) from the parallel approximation \( [q_{1c,asmp} \approx 0.035] \). As \( q_1 \) increases, the asymptotic approximation becomes poorer, particularly for the lower neutral curve. On the other hand, the agreement is not so good for the modes with \( n = -2 \) and \( n = -3 \), even near the critical swirl. We have also obtained the onset of absolute instabilities\cite{Batchelor} for the case \( n = -1 \), which is marked in Fig. 1(a) with an asterisk. It corresponds to \( q_{1c} \approx 0.239 \) and \( \omega_0 \approx -0.52 \). It must be noted that the results for the equivalent q-vortex are very similar. For instance, using \( Re_1 = 10^5, q_{1c} \approx 0.031 \) and \( \delta = 2398134.5 \) (\( \epsilon = 0.21823 \)) in \( (10)-(12) \), one obtains \( Re \approx 1664, W_0 \approx -60.09, \) and \( q \approx 1.953 \), and this value of \( q \) is very close to the critical value \( q_c \), found for that value of \( Re \) in the q-vortex.\cite{Batchelor} The similarity in the local stability results between Batchelor and q-vortex is due to the fact that \( q_1 \) is small, though \( q \) is not (this is the reason why we look here for viscous-centre modes, which are the only unstable ones for these high values of \( q \)).\cite{Batchelor, Batchelor2} So that the local velocity profiles of Batchelor’s vortex at \( z = 1 \) are close to the velocity profiles for the corresponding q-vortex \( (9) \). However, as we shall see below, this agreement in the near-parallel, local stability properties of Batchelor’s vortex and q-vortex is dramatically destroyed as one follows the stability of Batchelor’s vortex along \( z \) with the PSE.

To start the integration with the PSE we have selected some convectively unstable values of \( q_1 > q_{1c} \) in Fig. 1. In particular, we have chosen the values of the swirl parameter \( q_1 = 0.0325, 0.04, 0.06, \) and \( 0.1 \) which are unstable in some frequency range for \( n = -1 \) in the near-parallel approximation. The last three values of \( q_1 \) are also unstable for \( n = -2 \) and \( n = -3 \). The dispersion relations \( \gamma(\omega) \) and \( \alpha(\omega) \) for several of the less stable modes obtained with the near-parallel approximation \( (28) \) in these cases are plotted in Figs. 2-7. It is observed that for \( q_1 \geq 0.06 \) there exist six or more unstable modes in some frequency ranges. Curiously, all these plotted modes, both the stable and the unstable ones, have very similar group velocities \( c_g \equiv \partial \omega / \partial \alpha \), that practically coincide with the phase speeds because all the functions \( \alpha(\omega) \) collapse in almost the same straight lines [see Figs. 2(b)-7(b)].

Figures 8-16 summarize the results obtained from the PSE for the most unstable modes at different frequencies for which the azimuthal mode \( n = -1 \) is unstable according to the near-parallel approximation. The numerical integration of the PSE \( (18) \) is started at
z = 1 with the local eigenvalue and eigenfunction of the NP approximation (28). We have used several values of δz in the numerical simulations, ranging between 0.0001 and 0.004, checking that the results are practically independent of δz, provided that it is large enough to avoid the numerical instabilities.\textsuperscript{19,24} We have used several values of the numerical stabilization parameter defined and discussed in Ref.\textsuperscript{24} (this parameter is always larger than δz/2). It is observed in Figs. 8-16 that the results for the growth rate γ(z) obtained from the PSE and from the local near-parallel stability equations are quite different in most of the cases considered for n = −1, which is the azimuthal mode that first becomes unstable. The results are reasonably similar at the initial stages just downstream z = 1 (note that the abscissas in Figs. 8-16 is not z, but $Z \equiv z/\Delta = zRe_1/4$; i.e., the axial length made dimensionless with $r_c$ instead of $z_c$). Initially, both sets of stability equations yield the same stabilization trend as z increases. Eventually, however, the difference increases abruptly, and the mode tracked with the PSE becomes suddenly stable, while the mode obtained locally with the NP approximation remains unstable for a much longer axial distance. This feature is very marked at, or very near, the most unstable frequencies for each $q_1$, especially when $q_1$ is sufficiently larger than $q_{i1}^{c1}$, and the initial growth rate is not so small. For instance, at $\omega = −0.6$ for $q_1 = 0.06$ (Fig. 14), and at $\omega = −0.61$ for $q_1 = 0.1$ (Fig. 16): In the first case, the most unstable mode at z = 1 for n = −1 becomes stable at $Z \simeq 3125$ with the PSE, while it remains unstable up to $Z \simeq 14500$ when using the local NP approximation (Fig. 14); in the second case, the mode obtained with the PSE becomes stable at $Z \simeq 10065$, while it remains unstable for $Z > 30000$ when using the local NP approximation (Fig. 16). Thus, nonparallel effects due to the axial evolution of the perturbations becomes more important as one moves downstream of the initial axial location, stabilizing the vortex in an abrupt way much faster than the prediction of the local stability analysis, especially close to the most unstable frequencies for each value of the swirl parameter $q_1$ considered. The physical explanation of this behavior is possibly related to the associated abrupt increase of the wave number $\alpha$ for these unstable modes when computed from the PSE [Figs. 8(b)-16(b)]: The interaction between the different modes along z eventually generates an abrupt decrease of the axial wavelength of the most unstable perturbation, producing a spatial detuning that may explain the abrupt stabilization. In the local NP approximation this effect is not possible because the axial evolution of the perturbations themselves is not taken into account. It should be mentioned that no significant differences are appreciated in the perturbations form (eigenfunctions) when computed from the NP approximation or from the PSE at these values of z. This behavior of the PSE results has been checked out by using different values of the step size $\delta z$.

For n = −2 and n = −3, the behavior is qualitatively different (see Figs. 17-20). The growth rate γ(z) from the PSE and from the NP approximation have similar stabilization trends, and the stabilization distances computed from both approximations are quite close to each other. This is so for all the values of $q_1$ and frequencies considered.

**IV. CONCLUSION**

The stabilization distance of trailing vortices predicted by a local, near-parallel approximation is much larger than that predicted from the PSE, which takes full account of non-parallel effects, for the high Reynolds numbers of interest in actual trailing vortices and for perturbations with n = −1, which is the first mode to become unstable as the
swirl parameter is increased. This feature is specially marked for the most unstable frequencies for each \( q_1 > q_{1c} \) at a given Reynolds number. The axial evolutions of unstable perturbations with \( n = -2 \) and \( n = -3 \) are however quite similar when computed from the local near-parallel approximation and from the PSE, being the differences in the computed stabilization distances small when compared to the mode \( n = -1 \). Thus, the present study is a warning on the use of parallel-flow approximations for the prediction of stabilization distances in actual trailing vortices with axial flow.

**Acknowledgments**

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FIG. 1: Neutral curve in the \((\omega, q_1)\)-plane for \(Re_1 = 10^5\) and \(n = -1\) (a), \(n = -2\) (b), and \(n = -3\) (c), computed with the near-parallel approximation (28), compared with the parallel asymptotic approximation (35) (circles). The dashed line corresponds to the location of the maximum growth rate, which, for \(n = -1\) (a), ends at the conditions for the onset of absolute instability (asterisk).\(^{10}\)
FIG. 2: Dispersion relations $\gamma(\omega)$ (a) and $\alpha(\omega)$ (b) for several of the less stable modes with $n = -1$ for $\Re = 10^5$ and $\Im = 0.035$, computed with the near-parallel approximation (28).

FIG. 3: As in Fig. 2, but for $\Re = 0.04$. 

*
FIG. 4: As in Fig. 2, but for $q_1 = 0.06$.

FIG. 5: As in Fig. 2, but for $q_1 = 0.1$. 
FIG. 6: Dispersion relations $\gamma(\omega)$ (a) and $\alpha(\omega)$ (b) for several of the less stable modes with $n = -2$ for $q_1 = 10^5$ and $q = 0.06$, computed with the near-parallel approximation (28).

FIG. 7: As in Fig. 6, but for $n = -3$. 
FIG. 8: Growth rate $\gamma$ (a) and wavenumber $\alpha$ (b) as a functions of $Z = z/\Delta = zRe^{1/4}$ from the PSE (lines) and from the local, near-parallel stability equations (dots) for several modes corresponding to $n = -1$, $\omega = -0.31$, $Re = 10^6$, in a Batchelor vortex with $\lambda_1 = 0.0325$.

FIG. 9: As in Fig. 8, but for $\omega = -0.36$. 
FIG. 10: As in Fig. 8, but for $\omega = -0.26$.

FIG. 11: As in Fig. 8, but for $q_1 = 0.04$ and $\omega = -0.39$. 
FIG. 12: As in Fig. 8, but for $q_1 = 0.04$ and $\omega = -0.29$.

FIG. 13: As in Fig. 8, but for $q_1 = 0.04$ and $\omega = -0.49$. 
FIG. 14: As in Fig. 8, but for $q_1 = 0.06$ and $\omega = -0.6$.

FIG. 15: As in Fig. 8, but for $q_1 = 0.06$ and $\omega = -0.4$. 
FIG. 16: As in Fig. 8, but for $J_1 = 0.1$ and $\omega = -0.61$.

FIG. 17: Growth rate $\gamma$ (a) and wavenumber $\alpha$ (b) as a functions of $Z$ from the PSE (lines) and from the local, near-parallel stability equations (dots) for several modes corresponding to $n = -2$, $\omega = -0.84$, $J_1 = 10^5$, in a Batchelor vortex with $J_1 = 0.04$. 
FIG. 18: As in Fig. 17, but for $q_1 = 0.06$ and $\omega = -1.26$.

FIG. 19: Growth rate $\gamma$ (a) and wavenumber $\alpha$ (b) as a functions of $Z$ from the PSE (lines) and from the local, near-parallel stability equations (dots) for several modes corresponding to $n = -3$, $\omega = -1.36$, $Re_1 = 10^5$, in a Batchelor vortex with $q_1 = 0.04$. 
FIG. 20: As in Fig. 19, but for $q_1 = 0.06$ and $\omega = -1.92$. 
**AST4-CT-2005-012238**

**FAR-Wake**
Fundamental Research on Aircraft Wake Phenomena

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Specific Targeted Research Project

Start: 01 February 2005
Duration: 40 months

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**Experiments on vortex meandering**

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1 Introduction

Work Package 1 of the projet FAR-Wake deals with the dynamics and instabilities of single vortices and multiple-vortex systems. Task 1.1, entitled “Waves on vortices” focusses on phenomena related to the dynamics of a single vortex. The present study is part of Subtask 1.1.1 on “Vortex meandering”.

In this work, the dynamics of a single vortex generated by a generic wing in a water channel are investigated. The objective is to obtain detailed qualitative and quantitative information concerning vortex meandering, that will allow a comparison and confrontation with the different theoretical analyses performed in the same Subtask by partners UPS-IMFT and UMA. Such comparisons are expected to shed further light on the physical origin of this previously unexplained phenomenon.

2 Facilities and setup

2.1 The water channel

The facility used for the experiments is a recirculating water channel with a free surface. It has a test section of dimensions 37 cm (width) × 50 cm (height) × 150 cm (length). The free stream velocity $U_{\infty}$ can be chosen in the range 5–100 cm/s by setting the pump rotation frequency $F_p$. The turbulence intensities associated with the streamwise and transverse velocity components are approximately 1.5% and 0.6%, respectively. The bottom and side walls of the test section are made out of glass. In addition, a glass window was inserted downstream of the test section on the wall normal to the stream, thus allowing visual access to the flow inside the test section from five different directions. A scheme of the test section can be found in figure 1.

2.2 The wing

In order to generate a single vortex, a rectangular NACA0012 half-wing with a chord $c = 10$ cm was placed in the upstream part of the test section. It was manufactured out of polyvinyl chloride and has a span of 15 cm. The edge at the wing tip was rounded, with a varying diameter equal to the local thickness of the wing. The wing was mounted vertically on a profiled U-frame positioned along the side and bottom walls of the channel (see figure 2), so that the tip reaches the middle of the channel cross section. The tip of the trailing edge was chosen to be the origin of the frame of reference defined in figures 1 and 2, is the streamwise direction, and are the horizontal and vertical transverse directions. In this frame, we denote $x$, $y$ and $z$ the three components of the velocity. It was possible to adjust the angle of attack $\alpha$ manually by rotating the wing around an axis parallel to the (vertical) span direction. A small pipe network was machined inside the U-frame and the wing to allow for the injection into the flow of fluorescent dye (aqueous solution of fluorescein) close to the wing tip, in order to visualize the lift-generated trailing vortex.
2.3 Visualization setup

An argon ion laser (model Stabilite 2017 from Spectra Physics) was coupled to an optical fibre to illuminate the dye injected into the flow in a vertical plane, from the bottom to the top of the test section. Several injection holes in the wing were tested to find the one for which the dye remains in a region as close as possible to the center of the vortex. Taking advantage of the downstream channel window, a monochrome high speed video camera (Phantom V5 from Vision Research) was positioned for viewing along the \( O_z \) direction, in order to capture cross-sectional images of the vortex (see figure 1). All visualisations were made with the laser light sheet located at a distance \( z/c = 1.1 \) from the trailing edge of the wing.

2.4 Stereoscopic Particle Image Velocimetry (stereo-PIV) setup

A stereo-PIV technique was developed, in order to measure the three-dimensional velocity field of the wing tip vortices. A Nd-YAG pulsed laser was positioned underneath the test section at the desired distance from the wing, generating a 3 mm thick vertical light sheet. The flow was seeded with silver-coated particles (Dantec), whose size (100 \( \mu \)m diameter) was small compared to the characteristic scale of the vortex (diameter typically of the order of 1 cm). Images were taken with two high-resolution digital cameras (Roper Redlake, 4000\( \times \)2672 pixels\(^2\)), each viewing the laser sheet along a direction forming a 30\(^\circ\) angle with the axis of the channel (see figure 1). In order to reduce the distortions arising when looking through a liquid layer, the orthogonality between the cameras’ lines of sight and the air–liquid interface was established by placing water-filled prisms between the object plane and the recording lenses. Special care was taken to enforce the so-called Scheimpflug condition, which requires the object plane, the lens plane, and the image plane to intersect on a single line. This ensures a good focus on the object plane by diminishing the depth-of-field restrictions. An overview of stereo-PIV techniques, coupled with the use of prisms and Scheimpflug mounts, can be found in Prasad & Jensen [1] and Zand & Prasad [2].

The computing algorithm used for the extraction of fluid displacement from particle image pairs is based on a 2–D cross-correlation PIV code developed by Meunier & Leweke [3], which has been successfully used in previous experimental studies on vortex flows [4, 5].

3 Results

In order to investigate the dependence of vortex meandering on Reynolds number and axial velocity, nine flow configurations were studied, characterized by different combinations of the pump frequency \( ( p = 25 \text{ Hz}, 35 \text{ Hz}, 45 \text{ Hz}) \) and the angle of attack of the generating wing \( (\alpha = 6^\circ, 9^\circ, 12^\circ) \). For each case, stereo-PIV measurements were performed at a downstream location \( z/c = 1.1 \), in order to extract the parameters describing the vortex flow. Using dye visualisations made at the same position, it was then possible to follow the position of the vortex center over a period of time and extract statistical data characterizing vortex meandering. Proper Orthogonal Decomposition (POD) analysis was also performed, in order to obtain the most energetic modes of the vortex motion. Subsequent spectral analysis yielded information about the frequencies and the corresponding characteristic wavelengths of the vortex deformation.

3.1 Base flow characterization

3.1.1 Stereo-PIV measurements

Stereo-PIV measurements were performed to measure three-component velocity fields. For each configuration, 400 such fields were computed, from image pairs recorded at a frequency of 0.5 Hz. Each field was then fitted to the velocity field of a Gaussian vortex, in order to localism the coordinates of the vortex center, which are among the fitting parameters. (This Gaussian fit was here only used for the purpose of center finding; the actual shape of the velocity profile is discussed in the following section below.) The mean position of all fields was computed, and each individual field was then translated, so that the vortex was always located at the mean position. The velocity components were then averaged over the 400 fields. The process of ‘recentering’ each vortex allows a correct estimation of the mean vortex velocity field, since the lateral motion due to meandering does not interfere with the averaging.
process. Not correcting for the vortex motion would have lead to an underestimation of the vorticity and axial velocity defect at the center, and to an overestimation of the vortex radius.

Figures 3, 4 and 5 present the two-dimensional distributions of streamwise vorticity and axial velocity obtained for the nine configurations tested. Although the vortex is mainly axisymmetric, the wake of the wing can be observed in the axial velocity fields. A major source of error arises from the presence of bubbles appearing at the center of the vortex for high values of the Reynolds number. The low pressure at the center attracts air bubbles from the outside region. These bubbles appear on the frames captured by the cameras, forming a bright horizontal line (in the \( \text{Ox} \) direction) going through the center of the vortex. This leads to erroneous values of the vorticity in a localized region around \( y = 0 \), since \( y \) is underestimated there. This is the case for all \( \alpha \) at the highest pump speed \( (F_p = 45 \text{ Hz}) \) and also for \( \alpha = 12^\circ \) at \( F_p = 35 \text{ Hz} \).

3.1.2 Vortex velocity profiles

A standard way to characterize a vortex is to compute the azimuthally averaged profiles of swirl and axial velocity components as function of the distance \( r \) from the vortex center. Comparison with analytical vortex models, using least-square fits, then allows the determination of the characteristic vortex parameters. For the cases with erroneous \( y \) measurements mentioned above, the velocities were not averaged over the full azimuth, but only on a vertical line (\( \text{Ox} \) direction) going through the center of the vortex. There, \( x \) is the only velocity component taken into account for the azimuthal velocity, allowing a correct estimation of this profile.

**Axial velocity defect** The freestream velocity \( U_\infty \) was obtained by taking the maximum of the two-dimensional axial velocity profile. The axial velocity defect is defined as the difference between \( U_\infty \) and the local axial velocity in the channel. The azimuthally averaged radial profile of this quantity, \( \langle \rangle \), represents the axial velocity profile in the frame of reference moving with the vortex. It was found that for the vortices generated with the present set-up, this profile could be very well fitted by a Gaussian distribution given by

\[
\langle \rangle = r=0 \cdot \left(-\frac{r}{a_w}\right)^2
\]

where \( r=0 \) is the maximum axial velocity defect, measured at the center of the vortex. Using this value, least-square fitting of the measured profiles to (1) allows determination of the radius \( a_w \) characterizing the size of the axial velocity distribution. Figures 6–8 show in their right columns the measured profiles of axial velocity, as well as the Gaussian fits, for all configurations tested. The overall agreement is very good. The principal cause of mismatch is linked to the fact that the axial velocity field is not exactly axisymmetric outside the vortex core. For large radii, the presence of the wing wake has a major effect on the axisymmetric average; it is responsible, e.g., for the bump observed in figure 8(d).

**Azimuthal velocity** The total circulation \( \Gamma \) of the vortex was computed by integrating the cross-sectional velocity on a closed contour surrounding the entire field of view of the PIV measurement, as shown in figures 3–5, in order to take into account as much vorticity as possible. An approach similar to the analysis of the axial velocity, using a Gaussian fit, was first tried on the azimuthally averaged vorticity profile, but it turned out not to be satisfactory. Instead, following the approach of Fabre & Jacquin [6], the azimuthal velocity profile \( \theta \) was fitted with their VM2 vortex model, according to:

\[
\theta = \frac{\Gamma}{2} \cdot \frac{\phi-1}{\phi+1} \cdot \left[ 1 + \left( \frac{\phi}{1} \right)^4 \right]^{(1+\phi)/4} \left[ 1 + \left( \frac{\phi}{2} \right)^4 \right]^{(1-\phi)/4}
\]

In this model, two radii \( a_1 \) and \( a_2 \) are defined. \( a_1 \) corresponds to the radius up to which \( \theta(\cdot) \) can be considered roughly linear in \( r \), and \( a_2 \) is the minimum radius of a disk centered on the vortex inside which all the vorticity is contained. A third parameter quantifies the evolution of \( \theta \) between \( a_1 \) and \( a_2 \). This model was proposed based on theoretical arguments concerning the structure of trailing vortices. Figures 6–8 (left column) show that the match with the measured velocity profiles is very good.
From the dimensional quantities \( U_\infty, \Gamma, a_1, a_2, \phi, W_r=0 \) and \( w \), obtained directly or by fitting from the measured velocity fields, the following non-dimensional parameters were calculated:

- the chord-based Reynolds number \( cU_\infty/\nu \),
- the vortex Reynolds number \( \text{Re} = |\Gamma|/\nu \),
- the axial flow parameter (inverse swirl parameter) \( \phi = 2(1_{\Gamma=0}/\Gamma) \),
- the ratio \( a_2/a_1 \) of external and internal radii of the vorticity distribution,
- the ratio \( a_w/a_1 \) between the radial scales of the axial velocity and the vorticity profiles.

The values of all dimensional and non-dimensional parameter for all nine configurations tested are listed in table 1.

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Table 1: Vortex parameters for all configurations tested.

The linear dependence of the total circulation \( \Gamma \) versus the angle of attack \( \alpha \) is presented in figure 9. A slight circulation drop is observed for the highest angle \( \alpha = 12^\circ \), and in particular for \( \beta = 25 \text{ Hz} \), for which \( U_\infty \) is the smallest. This marks the limit between the linear regime and the stall regime, which, as shown below, can also be seen in the measurements of the amplitude of vortex meandering.

3.2 Statistical analysis of vortex position

In order to track the lateral vortex position in the flow, dye visualizations were performed as described in section 2.3. For each flow listed in table 1, 16000 frames were taken at a rate of 300 Hz over a period of 54 seconds. An example of a visualization frame is shown in figure 10. The high acquisition rate allows a study of phenomena occurring over a wide range of frequencies. In this study, we restrict our analysis to low frequencies, focusing only on the long-wavelength (low-frequency) meandering of the vortex. On a given frame, the coordinates \((x_c, y_c)\) of the vortex center were obtained by computing the “center of mass” based on the brightness (light intensity) of the image pixels. This approximation is based on the reasonable assumption that the dye distribution is centered on the vortex (symmetric with respect to the center), which depends on the adequate positioning of the dye injection holes on the wing. Figure 10 shows that this condition is satisfied fairly well.

In order to characterize the distribution of the transverse position of the vortex center, the eigenvalues \( \lambda_M \) and \( \lambda_m \) of the covariance matrix of \( x_c \) and \( y_c \) were computed. If we impose
The two corresponding eigenvectors $\mathbf{v}_M$ and $\mathbf{v}_m$ give the directions in which the statistical dispersion is maximal and minimal, the eigenvalues being the corresponding variances in both directions. $M$ and $m$ can be considered as dispersion radii in the $\mathbf{v}_M$ and $\mathbf{v}_m$ direction. An angle $\beta$ can be defined between the horizontal direction and the direction of the major axis $\mathbf{v}_M$ of the vortex position distribution. A convenient way to illustrate these quantities is to plot the ellipse of radii $M$ and $m$ aligned in the principal directions $\mathbf{v}_M$ and $\mathbf{v}_m$. This is done in figures 11 and 13 for all configurations. For comparison, figure 11 also shows the vortex positions obtained from each individual frame.

The values of $M$, $m$ and $\beta$ are listed in table 1. Whereas the mean amplitude of displacement (given by $M$ and $m$) remains well below the inner vortex core radius at this downstream distance (11.2 chords), large excursions exist, too, which frequently exceed two core radii. For low values of Re, the principal direction of the distribution is close to the vertical axis. It corresponds to the direction of alignment of the wing. No convincing explanation for this correlation can be put forward at this point. As Re and $\beta$ increase, decreases, tilting the distribution towards a direction normal to the wing. This rotation of the distribution with increasing Re is consistent with the observations of Devenport, Zsoldos & Vogel [7], although they did not compute the variances in the same way.

Figure 12 shows the evolution of $M$ and $m$ with $\alpha$. These radii lie in the ranges $0.4 \leq M \leq 0.75$ and $0.26 \leq m \leq 0.42$, i.e. the characteristic amplitude of meandering at this downstream location is roughly 50% of the viscous inner core radius. The general behavior of the radii is to decrease with $\alpha$. An exception to this rule is observed for $\alpha = 12^\circ$ and $p = 25$ Hz. In this case, where the angle of attack is high and the velocity is low, the stall regime is reached. Zaman et al. [8] showed that low frequency oscillations can be observed in the flow over an airfoil near stalling conditions. This phenomenon may lead to an additional perturbation of the vortex, resulting in higher vortex meandering amplitudes.

In order to estimate the statistical convergence of our results, figure 14(a) shows the evolution of the second moment of $M$ with the number of frames used. For all configurations, gradually decreases and approaches an asymptotically constant limit. The slope $\varepsilon$ of these curves is plotted in figure 14(b). For the sample size of 16000 frames, we find $\varepsilon = 4 \cdot 10^{-4}$ cm/(1000 frames).

### 3.3 Analysis of the vortex perturbation

The previous section focussed on the overall position of the vortex. In this section, the precise nature of the vortex perturbation, with respect to an axisymmetric straight reference flow, is investigated. First, the spatial structure of the perturbations will be described by determining the most energetic modes. Then, focussing on the first most energetic mode, corresponding to a lateral displacement, characteristic frequencies and wavelengths will be extracted from the measurements and compared to theoretical predictions.

#### 3.3.1 Mode calculation

A common way to extract a set of modes characterizing the perturbation of a given base flow is to perform a singular value decomposition (or Proper Orthogonal Decomposition – POD [9]) using a series of flow fields. In the case of a vortex, it is appropriate to use the vorticity distribution for this type of analysis.

The procedure is illustrated for the configuration with $\alpha = 6^\circ$ and $p = 25$ Hz ($\text{Re} = 8700$, $\text{p} = 0.27$). The video camera was positioned as shown in figure 1 to perform high-speed planar (2D) PIV measurements at $F = 11.2$. For a duration of 8 seconds, 16000 particle images were recorded at an acquisition rate of 2000 Hz. The POD was then performed on the whole series of vorticity fields computed from these images. Through the singular value decomposition of a matrix containing all the fields, the perturbations were developed into an ordered set of orthogonal vectors (modes). The projection of the entire data onto the first vector has the highest energy (or variance), and this variance decreases with increasing order of the vectors. For the example chosen here, the relative energy of the first 20 modes is presented in figure 15(a). The corresponding spatial structure of the first 6 modes can be found in figure 16. (In this figure and others alike, all frames were plotted with a color scale centered on zero (black); since the frames represent eigenvectors, the notion of amplitude is not present.) The energy decrease is clearly visible but gets smaller with increasing mode.
number. The first mode can easily be linked to the mean (time-averaged) field. It represents an axisymmetric vortex. Modes 2 and 3, presented in figures 16(b) and 16(c), are centered on the vortex. They have an azimuthal symmetry of order 1, and represents a global lateral displacement of the vortex. The principal directions of the two modes form a $90^\circ$ angle. A linear combination of both modes is therefore sufficient to account for vortex displacements in all directions of the plane. It is possible to associate modes 2 and 3 with a so-called Kelvin wave of azimuthal wavenumber $m = 1$. (See Fabre et al. [10] and references therein for a description of Kelvin waves). The associated energy has similar values for the two modes. They form a high-energy doublet easily noticeable in figure 15(a). This means that a large part of the perturbation energy is used to displace the vortex, resulting in vortex meandering.

Mode 4 (figure 16(d)) has a second-order azimuthal symmetry. It describes an elliptic compression perturbation centered on the vortex. In terms of Kelvin waves, it can be interpreted as a combination of $m = 0$ and $m = 2$ waves. Modes 5 and 6 also have a second-order azimuthal symmetry. Both of them are similar to a Kelvin wave with $m = 2$. Nevertheless, contrarily to a pure $m = 2$ Kelvin wave, the center of the vortex is perturbed as well, revealing the influence of an $m = 0$ Kelvin wave. There is a 4 phase difference between the two. Similarly to modes 2 and 3, a linear combination of modes 5 and 6 results in a perturbation with the same structure, but rotated by an amplitude depending on the relative contributions of each initial mode. Any angle of rotation can be obtained. It is possible to recover pure $m = 0$ and $m = 2$ Kelvin waves by a linear combination of modes 4, 5 and 6. This is consistent with the fact that the order or magnitude of their associated energies is the same, compared to the rest of the modes. Note that an increase of the number of fields might lead to the decoupling of the Kelvin modes directly by the singular value decomposition.

Although it is possible, with this method, to find modes with even higher spatial complexity ($m = 3$, $m = 4$, etc.), such modes of lower energy are not considered further in this study, their impact being less important.

One important goal of this analysis is to extract a frequency characterizing vortex meandering, related to a characteristic wavelength of this phenomenon via the free stream velocity. The idea is to compute the projection of the complete flow field onto the most energetic displacement mode, in order to obtain a scalar temporal series. A spectral analysis can then be performed to reach the above objective. The frequency of meandering, related to a long-wavelength deformation convected through the measurement plane, is expected to lie in an interval around 1 Hz in our set-up (this was confirmed by results shown below). A relatively long time interval is required for a given measurement, in order to resolve the fluctuation spectrum correctly in this low frequency range. On the other hand, the determination of vorticity fields requires PIV measurements, involving acquisition of two frames closely spaced in time. Due to hardware constraints and the particular requirements for PIV measurements and the subsequent POD and spectral analysis for the present experiments, it was not possible to obtain accurate vorticity fields over a time interval long enough for the spectral analysis.

An alternative method was tested, in which POD decomposition was performed from dye visualizations, using the light intensity field of the frames instead of the vorticity field. Contrary to PIV measurements, dye visualization only requires a single frame at a time, allowing much longer acquisition periods. In order to demonstrate the validity of such an approach, a comparison was performed between the results obtained with the vorticity field and those obtained with the light intensity distribution.

During the acquisitions of the PIV images leading to the results for the vorticity field in figures 15(a) and 16, the flow also contained a small amount of dye (such as in figure 10), since some dye from a previous visualization experiment was leaking in through the open injection holes. In this way, the same frames could be used to computed the POD on the dye distribution, leading to a direct comparison with the vorticity modes. The energy distribution obtained from the dye is presented in figure 15(b), and the spatial structure of the modes can be found in figure 17. These figures are to be compared with figures 15(a) and 16. In figures 16 and 17, the modes are presented in the same order of energy content, and the axis scales are equal. The agreement between the two sets of results is extremely good, concerning the mode structure and spatial orientation. The fact that the sign is inverted for some modes is irrelevant, since the mode amplitude is arbitrary in this representation.

From this good agreement between the results obtained with vorticity and dye, we can draw the conclusion that the dye is a suitable marker of the flow structure, and that it can be used for POD analysis in the present case.
3.3.2 Meandering frequency and wavelength

For all configurations listed in table 1, the images used for calculating the statistics of the vortex center position in section 3.2 were used again for the perturbation mode decomposition and spectral analysis. As mentioned above, they were recorded with an acquisition rate of 300 Hz over a period of 54 seconds, which is long enough to resolve the fluctuations at low frequency correctly.

The results of this analysis are illustrated here for the case \( \alpha = 9^\circ \) and \( \beta = 35 \) Hz, i.e., \( \text{Re} = 18800 \) and \( \rho = 0.32 \). The six most energetic modes describing the perturbation of the vortex in this configuration are presented in figure 18, whereas figure 19 shows the associated eigenvalues. The overall picture is similar to the one described in section 3.3.1, and the same comments apply. One clearly observes the mean field, the two displacements modes, and the three modes coupling \( m = 0 \) and \( m = 2 \) Kelvin waves. Since the objective is here the characterization of the overall vortex motion, emphasis is put on the most energetic perturbation “vector”, the displacement mode 2 (figure 18(b)). The time-dependent projection of the visualization images on mode 2 is presented in figure 20 for a time interval of 7 seconds. As expected, the projection is centered on zero. It is possible to visually identify a period of approximately 1 second in this signal, synonymous to a 1 Hz frequency. It means that the vortex oscillates (on average) around the mean position (defined by mode 1, figure 18(a)) in the direction given by the orientation of mode 2.

Figures 22(c) and 22(d) present the power spectral density versus the frequency \( m \) for the time signal in figure 20; on both a linear and logarithmic scale. In this particular case, it is difficult to extract from these plots a dominant frequency with a precision better than the one obtained visually from figure 20. Nevertheless, it is possible to identify a frequency \( f_c \approx 1 \) Hz above which the power spectral density decreases abruptly, following a power-law decay of slope -3.5 between \( m = 1 \) Hz and \( m = 10 \) Hz.

This behavior is also observed for all the other configurations, as shown in figures 21–23. In some cases (e.g., figures 22(a) and 23(c)), a peak is clearly visible, the power spectral density decreasing rapidly below \( f_c \). For all configurations, \( f_c \approx 1 \) Hz appears to be a good estimation of the dominant meandering frequency at the fixed downstream position \( z/c \approx 11.2 \) in the present set-up. However, conclusions on the evolution of \( f_c \) with Re or \( \rho \) are difficult to draw, given the shape of the spectra obtained in this study.

The frequency \( f_c \) of vortex meandering measured at a fixed distance behind the wing can be related to the axial wavelength \( \lambda = 2 \) and wave number \( k = m/m \) of the vortex deformation in the frame of reference moving with the vortex, using transformations involving the free stream velocity \( U_\infty \): \( F_c = \omega \) and \( k = 2 \). Figure 24 shows the power spectral density as function of the non-dimensional wave number \( \omega \) for the three cases where a clear maximum can be identified in the spectrum. For all three, this maximum is located approximately at \( \omega = 0.05 \), corresponding to \( F_c \approx 120 \), i.e., the vortex displacement wavelength is more than two orders of magnitude larger than the inner vortex core radius, which represents indeed a very long-wavelength deformation. Similar values of \( F_c \) are also observed for all other cases studied.

3.3.3 Comparison with theory – transient growth and optimal perturbation

Antkowiak & Brancher [11] and Fontane, Brancher & Fabre [12] have investigated the possibility of transient energy growth in a Lamb–Oseen (Gaussian) vortex without axial flow, and its excitation by random fluctuations (stochastic forcing). They found that transient growth indeed exists in this generic vortex flow, and that for long observation times (compared to the vortex turnover period) a long-wavelength translation mode dominates.

Despite the fact that in the present experiments the vorticity profile is not Gaussian, and that there exists an axial core velocity, an attempt is made to compare the results presented in the preceding sections with those of Brancher and co-workers. For this, the non-dimensional time corresponding to our observations needs to be determined. In the theoretical analysis, time is non-dimensionalised by the angular velocity \( \Omega_o \) of the fluid at the vortex center, which is equal to half the vorticity there and given roughly by \( \Gamma (2 \pi / \Omega) \). In the experiments, it is the time it takes the fluid to move from the trailing edge of the wing to the measurement point.
plane: $t = z/U_\infty$. This leads to:

$$\tau = \Omega_0 t = \Gamma_2 \pi a_1 \cdot z U_\infty = \frac{1}{2} \pi \left( \frac{\Gamma_c U_\infty}{z_{c}} \right) \left( a_2 \right)^{-2}$$

(3)

With the non-dimensional measurement location $z/c = 1.2$ and the values of the other parameters given in Table 1, one finds that for all configurations tested the non-dimensional observation time is given by: $\tau = 150 \pm 10$.

For the Gaussian vortex without axial flow, Antkowiak & Brancher [11] have calculated, for each non-dimensional wave number $k a$, where $a$ is the vortex core radius, an optimal time $\tau_{\text{opt}}$, for which the transient vortex perturbation at this $k a$ is maximum. Figure 25 reproduces their diagram and extends it to include the present result (with $k a = 0.05$), obtained for a similar Reynolds number range. The agreement is very good, despite the differences in vortex structure, suggesting that these differences are probably not of high relevance in the context of vortex meandering. Figure 26, taken from [13] and showing the maximum gain of transient perturbations as function of wave number, also gives the structure of these perturbations. The low-wave number (large wavelength) branch close to the values measured in the experiments belongs indeed to a translation (or displacement mode) of the vortex, whose spatial structure at the optimal time is the same as the most energetic perturbation in the experimental observation, shown in figure 18(b).

This good agreement between the present experimental results and the theoretical work carried out by UPS-IMFT strongly suggests that the meandering phenomenon observed in the water channel is indeed the manifestation of an optimal growth of transient perturbations. The detailed measurements provided here would allow an even closer comparison with the transient growth predictions. A similar analysis to the one by Antkowiak & Brancher [11] could be carried out, but using the actual measured velocity field, including axial flow, and producing a diagram with the growth factor as function of wave number at the non-dimensional time corresponding to the experimental measurement.

4 Summary and conclusion

The goal of this work was to provide detailed experimental data about vortex meandering. For this purpose, a single trailing vortex was generated in a water channel, using a half-wing. Nine configurations were tested, involving different free-stream velocities and angles of attack. The circulation-based Reynolds number varied in the range 8700–31500, and the axial velocity defect at the vortex center was between 27% and 55% of the maximum swirl velocity.

Detailed measurements of the three-dimensional vortex velocity profiles were carried out at 11.2 chord lengths behind the wing, involving stereo-PIV and recentering of individual PIV fields before averaging. The swirl velocity in the cross-sectional plane was found to correspond closely to the profile of the VM2 vortex model proposed by Fabre & Jacquin [6], whereas the axial velocity profile was to a good approximation Gaussian. The fitting parameters were given for all configurations.

The first part of the meandering analysis dealt with the statistics of the vortex center positions in the measurement plane. It was found that the amplitude of vortex displacement decreases with Reynolds number, in agreement with Devenport’s [7] earlier findings. The meandering amplitude was found to be of the order of the core radius, and the principal directions of motion were identified.

In the second part, singular value decomposition (or Proper Orthogonal Decomposition - POD) analysis was carried out, using high-frequency vorticity data from PIV and dye visualisation images. After confirming that both of these inputs give similar results, long time series of dye visualizations were analyzed by POD. The most energetic perturbations were found to be displacement modes of the vortex. Spectral analysis of the projection of the time series on these displacement modes showed that they are characterized by low-frequency oscillations (at the fixed measurement position), corresponding to wavelengths in the frame of reference moving with the vortex that are two orders of magnitude larger than the vortex core radius.

Comparison was made with theoretical results concerning transient growth in a Gaussian vortex without axial flow, presented by Antkowiak & Brancher [11]. Despite the differences
in the vortex velocity profiles, good agreement was found, concerning mode structure and wavelength, between the present measurements and their optimal perturbation at the given observation time. This agreement strongly suggests that the meandering phenomenon observed in the water channel can be explained by the transient growth of vortex perturbations, initiated by background noise in the flow or by turbulence in the wake of the wing. It is likely that this conclusion remains valid for other experimental facilities of similar type, such as wind tunnels.

References


Figure 1: Schematic of the test section of the water channel, as seen from the top, showing the stereo-PIV and visualization set-ups. Glass windows are shown in blue. A laser (not represented) was placed underneath the channel, illuminating the flow in a vertical plane.

Figure 2: Wing set-up in the water channel.
Figure 3: Two-dimensional distributions of vorticity \([s^{-1}]\) (left column) and inverse axial velocity \([cm/s]\) (right column), for \(\alpha = 6^\circ\) and \(\rho = 25\ Hz, 35\ Hz\) and \(45\ Hz\) (top to bottom).
Figure 4: Same as figure 3, for $\alpha = 9^\circ$. 

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Figure 5: Same as figure 3, for $\alpha = 12^\circ$. 
Figure 6: Azimuthally averaged radial profiles of swirl velocity (left column) and axial velocity (right column) for $\alpha = 6^\circ$ and $f_p = 25$ Hz, 35 Hz and 45 Hz (from top to bottom). Circles represent experimental measurements, and lines are least-squares fits: VM2 model for $U_{\theta}$, Gaussian for $W$.  

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Figure 7: Same as figure 6, for $\alpha = 9^\circ$. 
Figure 8: Same as figure 6, for $\alpha = 12^\circ$. 
Figure 9: Variation of total vortex circulation with angle of attack. Circles, triangles and squares correspond to $\alpha = 6^\circ$, $9^\circ$ and $12^\circ$, respectively. Black, gray and white markers represent $f_p = 25$ Hz, $35$ Hz and $45$ Hz, respectively. The solid line represents a linear fit, yielding a slope $= 1.71$.

Figure 10: Example frame from the high-frequency dye visualization used for vortex position statistics and the singular value decomposition.
Figure 11: Distributions of the transverse vortex center positions at $z/c = 1.12$. Coordinates are centered on the mean position, and axes are in cm. $f_p = 25$ Hz, 35 Hz and 45 Hz (top to bottom) and $\alpha = 6^\circ, 9^\circ$ and $12^\circ$ (left to right). Axis units are cm. Each dot represents the position in one visualization frame. The red curve is an ellipse of radii $a_M$ and $a_m$, whose major axis forms an angle $\beta$ with the horizontal (see table 1).
Figure 12: Radii $a_M$ (solid symbols) and $a_m$ (open symbols) as function of $\alpha$. Squares, triangles and circles correspond to $f_p = 25$ Hz, 35 Hz and 45 Hz, respectively.
Figure 13: Ellipses associated with the distribution of the vortex centers for $\alpha = 6^\circ$, $9^\circ$ and $12^\circ$. Solid lines, squares and circles correspond to $p = 25$ Hz, 35 Hz and 45 Hz, respectively. Axis units are cm.
Figure 14: (a) The second moment of the radius as function of the number of frames used for the calculations. (b) The slope $\varepsilon$ as function of $n_f$. $\varepsilon$ is computed every 1000 frames. White, gray and black markers represent $\mu = 25$ Hz, 35 Hz and 45 Hz, respectively. Squares, triangles and circles correspond to $\alpha = 6^\circ$, $9^\circ$ and $12^\circ$.

Figure 15: Comparison of the distributions of the first twenty singular values computed from the vorticity fields ($S_v$), and from the dye intensity in the frames ($S_c$). $\alpha = 6^\circ$ and $\mu = 25$ Hz.
Figure 16: Modes computed by singular value decomposition of the vorticity field time series obtained by high-speed PIV for $\alpha = 6^\circ$ and $\nu = 25$ Hz ($\text{Re} = 8700$ and $\gamma = 0.27$). The six most energetic modes are shown.
Figure 17: Same as figure 16, but obtained from dye intensity time series.
Figure 18: Modes computed by singular value decomposition of the dye intensity time series for $\alpha = 9^\circ$ and $\nu = 35$ Hz ($\text{Re} = 18800$ and $W_0 = 0.32$). The six most energetic modes are shown.
Figure 19: Singular value distribution of the most energetic modes for $\alpha = 9^\circ$ and $F_p = 35$ Hz ($\text{Re} = 18800$ and $\vartheta = 0.32$).

Figure 20: Temporal evolution of the projection of the dye images on mode 2 for $\alpha = 9^\circ$ and $F_p = 35$ Hz ($\text{Re} = 18800$ and $\vartheta = 0.32$).
Figure 21: Power spectral density $P$ of the projection of the dye image series on mode 2, as function of the frequency $f_m$, for $\alpha = 6^\circ$ and $p = 25$ Hz, 35 Hz and 45 Hz (top to bottom). The slope of the dashed line is -3.5.
Figure 22: Same as figure 21, for $\alpha = 9^\circ$. 
Figure 23: Same as figure 21, for $\alpha = 12^\circ$. 
Figure 24: Power spectral density of the projection of the dye image series on the displacement mode 2, as a function of the non-dimensional wave number \( k a_1 = \frac{2\pi a_1 f_m}{U_\infty} \).

\[ \alpha = 6^\circ, \; p = 25 \text{ Hz} \; (Re = 8700, \; \phi = 0.27) \]

\[ \alpha = 9^\circ, \; p = 25 \text{ Hz} \; (Re = 12700, \; \phi = 0.31) \]

\[ \alpha = 12^\circ, \; p = 35 \text{ Hz} \; (Re = 22500, \; \phi = 0.36) \]
Figure 25: Optimal time as function of wave number. Results from Antkowiak & Brancher [11] for a Gaussian vortex without axial flow are compared to the present observations. In this figure, the Reynolds number is defined as $Re = \frac{\Gamma}{2\pi \nu}$.
Figure 26: Optimal gain as function of wave number, and associated perturbation structure, for a Gaussian vortex without axial flow. From Brancher et al. [13].