Temporal DNS on vortex bursting and assessment

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Temporal DNS of vortex bursting and assessment

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Abstract. This study focusses on the vortex bursting phenomenon by performing Direct Numerical Simulation (DNS). The approach consists to simulate the propagation of two front waves in opposite directions along a simple vortex. The vortex bursting was first studied numerically by Moet et al. [12], this document proposes an extension of their analysis. Several simulations were performed with various front wave magnitudes to determine a status of this phenomenon. The present analysis deals with the vortex structure, the time evolution of enstrophy but also with the generation of small structures. It is shown that this sudden and strong phenomenon depends on the waves magnitude, and the consequences of the two waves collision can lead to a drastic change of the vortex structure and eventually to a development of a helical instability. These results allowed to establish two states of the vortex bursting: the first one for small wave magnitudes $\epsilon < 0.4$ with slight affectations, and the second one for larger wave magnitudes characterised by important changes of the vortex properties.
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1 Introduction

A realistic aircraft wake composed of two symmetric counter-rotating vortices with axial velocity will naturally develop the Crow-instability. Moet et al. [12] have simulated this instability and observed helical instability and vortex bursting, which seem to be related to the reconnection process causing the generation of multiple pressure waves propagating along the vortex cores. Very few specific results are available on the vortex bursting phenomenon, which is an archetype of strong localized non-linear interaction.

The vortex bursting phenomenon is often explained as resulting from the strong interaction of two waves propagating in opposite directions. Moet et al. [12] have demonstrated, using numerical simulation (Direct Numerical Simulation and Large Eddy Simulation), how the collision of two axisymmetric pressure waves is responsible of vortex bursting. Their results showed an abrupt change of the internal structure of the vortex accompanied by a drastic increase of the vortex core radius. They also found that the propagation of pressure waves causes a generation of axial velocity in the vortex core, which can lead to the development of a helical instability [9].

The present study aims to bring results more detailed and to establish a status of this phenomenon using DNS simulations. Thus, the first part of this report is dedicated to the numerical framework, with a short description of the numerical tools and the simplified approach considered here. The second part deals with the numerical results of the vortex bursting dynamics, through the collision of two waves propagating in opposite directions. The global dynamics is studied, then a quantification of the phenomenon is proposed.
2 Numerical tool and initial condition

2.1 Numerical code NTMIX3D

The code used for these studies is NTMIX3D, which has been developed by the Centre de Recherche sur la Combustion Turbulente (CRCT) of the Institut Francais du Pétrole (IFP). The parallel code solves the Navier-Stokes equations for a 3D unsteady compressible flow on a regular or irregular Cartesian grid. The solver is devoted to Direct Numerical Simulations and Large-Eddy Simulations.

A nondimensional formulation of the Navier-Stokes equations is used and high accuracy of the solution is guaranteed by a 6th order compact scheme for the discretisation in space, and time integration is performed with a 3rd order Runge-Kutta method. The code has been run in parallel (MPI) on the CRAY XD1 computers of CERFACS.

Governing equations

The conservation of mass, momentum and energy of a three-dimensional unsteady compressible viscous flow is described by the Navier-Stokes equations. In a 3D Cartesian coordinate system \((x, y, z)\), the Navier-Stokes equations in conservative form can be expressed as follows

\[
\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}(f - f_v) + \frac{\partial}{\partial y}(g - g_v) + \frac{\partial}{\partial z}(h - h_v) = 0 \tag{1}
\]

The state vector \(W\) is defined as

\[
W = \begin{pmatrix}
\rho \\
\rho u \\
\rho v \\
\rho w \\
\rho E
\end{pmatrix}
\tag{2}
\]

with \(\rho\) the density, \(u, v, w\) are the 3D velocity components, \(p\) the pressure and \(E\) the total energy \((E = c_v T + 1/2(u^2 + v^2 + w^2))\). The convective fluxes \(f, g\) and \(h\) in the directions \(x, y\) and \(z\) are defined as:

\[
f = \begin{pmatrix}
\rho u \\
\rho u^2 + p \\
\rho uv \\
\rho w \\
u(\rho E + p)
\end{pmatrix}, \quad
g = \begin{pmatrix}
\rho v \\
\rho uv \\
\rho v^2 + p \\
\rho vw \\
v(\rho E + p)
\end{pmatrix}, \quad
h = \begin{pmatrix}
\rho w \\
\rho wu \\
\rho w^2 + p \\
\rho vw \\
w(\rho E + p)
\end{pmatrix} \tag{3}
\]

The viscous fluxes can be expressed as:
2 Numerical tool and initial condition

\[ f_v = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u_j \tau_{xj} - q_x \end{pmatrix}, \quad g_v = \begin{pmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u_j \tau_{yj} - q_y \end{pmatrix}, \quad h_v = \begin{pmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u_j \tau_{zj} - q_z \end{pmatrix} \]  

(4)

where \( q_i \) are the components of the heat flux vector determined by Fourier’s law of heat conduction. Furthermore, \( \tau_{ij} \) is the Reynolds stress tensor linked to viscosity as postulated in Stokes’ hypothesis. Sutherland’s viscosity law is used to represent the variation of viscosity with temperature. The pressure is linked to the state vector by the equation of state for a perfect gas.

In NTMIX3D a nondimensional formulation of the Navier-Stokes equations has been used. This formulation is based on reference quantities \( (L_{\text{ref}}, a_{\text{ref}}, \rho_{\text{ref}}, \nu_{\text{ref}}) \) in a manner that the nondimensional quantities (denoted by \( * \)) are defined as

\[
\begin{align*}
  u_i^* \cdot a_{\text{ref}} &= u_i \\
  t^* \cdot \frac{L_{\text{ref}}}{a_{\text{ref}}} &= t \\
  \rho^* \cdot \rho_{\text{ref}} &= \rho \\
  \nu^* \cdot \nu_{\text{ref}} &= \nu
\end{align*}
\]

(5) – (8)

where \( a_{\text{ref}} \) is usually the speed of sound.

Numerical Methods

The nondimensionalized Navier-Stokes equations are discretized in space using a finite difference method. The computational meshes used for the type of simulations presented in this report are Cartesian, irregular in the horizontal and vertical transverse directions and regular in the axial direction.

The time integration has been done by a 3rd-order 3-stage Runge-Kutta method. When one considers \( y \) the solution of Cauchy’s problem \( y' = f(t, y) \), the integration following Runge-Kutta’s method gives

\[
\begin{align*}
  y(t + \Delta t) &= y(t) + \Delta t \cdot \hat{f}(t, y) \\
  \hat{f}(t, y) &= 1/4 \ K_1 + 3/4 \ K_3 \\
  K_1 &= f(t, y) \\
  K_2 &= f(t + \Delta t/3, y + \Delta t/3 \ K_1) \\
  K_3 &= f(t + 2\Delta t/3, y + 2\Delta t/3 \ K_2)
\end{align*}
\]

(9)

The maximal timestep is given by the limit of stability, expressed usually as Courant-Friedrichs-Lewy number CFL for the convective part and as the Fourier number for the
diffusive part. These numbers may be expressed by the dimensionless variables as (in one dimension)

\[ \text{CFL} = \text{Max}(\left(\frac{u^* + \sqrt{C_p^*(\gamma - 1)T^*}}{\Delta x^*}\right), \frac{\Delta t^*}{\Delta x^*}) \]  \tag{10} \\
\[ F\phi = \text{Max}(\frac{\mu^*}{\rho^*}, \frac{\Delta t^*}{\Delta x^*}, \frac{1}{Re}) \]  \tag{11} 

For the case studied \( \text{CFL} < 0.5 \) and \( F\phi < 0.1 \) appeared to be sufficient to provide numerical stability. Subsequently, for each iteration the time step is determined by the most strict of either condition over the entire mesh.

For regular grids the spatial derivatives of order 1 (convective terms) and order 2 (diffusive terms) are computed using a 6\textsuperscript{th} order compact scheme [8] (Padé type). For the first order derivatives \((du/dx)(x_i)\) of variable/function \(u\) at a point \(x_i\), the approximation \(u'_i\) is introduced, obtained by a Padé scheme which in its general form reads

\[
\begin{align*}
\beta u'_{i-2} + \alpha u'_{i-1} + u'_i + \alpha u'_{i+1} + \beta u'_{i+2} = \\
\frac{c}{6h}u_{i+3} - \frac{u_{i-3}}{6h} + b \frac{u_{i+2} - u_{i-2}}{4h} + a \frac{u_{i+1} - u_{i-1}}{2h}
\end{align*} \tag{12}
\]

In order to obtain a scheme with a truncation error of the order \(O(h^6)\), one has to satisfy the following constraint

\[ a + 2^4b + 3^4c = 2 \frac{5!}{4!}(\alpha + 2^4\beta) \]  \tag{13} 

The determination of the first order derivatives reduce to a linear pentadiagonal system with the following form

\[ Au' = Bu \]  \tag{14} 

Solving this system provides the approximation \(u'_i\) in every point. In the particular case with \(\beta = 0\), which is the choice made for NTMIX3D, one obtains a family of schemes that are associated to tridiagonal systems. The other constants of the scheme are \(\alpha = 1/3, a = 14/9, b = 1/9\) and \(c = 0\). Subsequently, the tridiagonal system is inversed by a Thomas algorithm. The prime interest for the use of such schemes is that one obtains high-level accuracy \((O(h^6))\) with a reduced stencil (5 points). Moreover, Lele [8] shows that his schemes possesses very weak dispersive errors (quasi-spectral behavior). In particular, the treatment of short-wavelengths by this type of schemes is remarkable. The treatment of boundary conditions brings about the deterioration of the accuracy at the edges of the computational domain \((O(h^3)\) at the boundary). Upwind compact schemes can be used in the vicinity of an edge in the case where nonperiodic boundary conditions are used. For the second order derivatives, a 6\textsuperscript{th}-order tridiagonal scheme is also used.
Nonuniform mesh

Nonuniform mesh treatment is required for applications that demand a high resolution of the mesh in a particular region of the computational domain. The use of irregular meshes is a solution in order to avoid an enormous amount of grid points. Several methods exist to perform derivative computations on variable grids such as the Jacobian transformation (JT) method and the method of Fully Included Metrics (FIM), see Gamet et al. [4]. The latter method has been implemented in NTMIX3D and has been used to perform the wake vortex simulation. The reason why the FIM method has been retained instead of the JT method is that the JT method can lead to large errors in the case of non-smoothly varying mesh spacings. In NTMIX3D the FIM method has been applied such that a fourth (respectively third) order approximation is obtained for the first (respectively second) derivative in space in the regions where the grid is stretched.

The FIM method involves direct inclusion of the metrics in the coefficients of the compact derivatives matrices to treat nonuniform meshes. This means adapting the original compact scheme, which is designed for uniform meshes. The main constraint that is imposed is that the obtained scheme for nonuniform meshes reduces exactly to the scheme for uniform meshes in case of a uniformly spaced grid.

In the following only the derivation of the scheme for the first derivative in space is described, as the derivation of the second derivative can be done in a similar manner (for details see Gamet et al. [4]).

The scheme for the first derivative in space in a general way (similar to Eq. (12)) is given by

\[ \alpha_i f_{i-1}' + f_i' + \beta_i f_{i+1}' = A_i f_{i+1} + B_i f_{i-1} + C_i f_{i+2} + D_i f_{i-2} + E_i f_i \]  

where the coefficients \( \alpha_i, \beta_i, A_i, B_i, C_i, D_i \) and \( E_i \) are functions of the nonuniform mesh spacings \( \Delta_k = x_k - x_{k-1} \). Following Lele [8], the relation between the former coefficients can be derived by matching the Taylor series coefficients of various orders. The truncation error is determined by the first unmatched coefficient in the Taylor series expansion. The following relations are obtained

\[ A_i + B_i + C_i + D_i + E_i = 0 \] (order 0)

\[ h_{i+1} A_i - h_i B_i + (h_{i+2} + h_{i+1}) C_i - (h_i + h_{i-1}) D_i = 1 + \alpha_i + \beta_i \] (1st order)

\[ h_{i+1}^2 A_i + h_i^2 B_i + (h_{i+2} + h_{i+1}) C_i = (h_i + h_{i-1})^2 D_i = \frac{2}{1!} (h_{i+1} \beta_i - h_i \alpha_i) \] (2nd order)

\[ h_{i+1}^3 A_i - h_i^3 B_i + (h_{i+2} + h_{i+1})^2 C_i = (h_i + h_{i-1})^3 D_i = \frac{3}{2!} (h_{i+1}^2 \beta_i + h_i^2 \alpha_i) \] (3rd order)

\[ h_{i+1}^4 A_i + h_i^4 B_i + (h_{i+2} + h_{i+1})^3 C_i = (h_i + h_{i-1})^4 D_i = \frac{4}{3!} (h_{i+1}^3 \beta_i - h_i^3 \alpha_i) \] (4th order)


If one wants to obtain a fourth order scheme, the solution can be found in a linear system composed of the five equations given in Eq. (16), where $A_i$, $B_i$, $C_i$, $D_i$ and $E_i$ the unknowns. The coefficients are given in a general form in the work of Gamet et al. [4]. In the expressions for the unknowns, the parameters $\alpha_i$ and $\beta_i$ are considered constants equal to their values for uniform meshes, namely $\alpha_i = \beta_i = 1/3$. This implies that the scheme for the first derivative in space exactly reduces to the scheme described previously for uniformly spaced grids. In case the grid does not vary smoothly the accuracy of the scheme reduces to fourth order.

For nonperiodic boundaries, Eq. (15) can not be applied to the points close to the boundary. Therefore, boundary schemes are required at points 1, 2, $N - 1$ and $N$. The scheme for the first derivative in space at point 1 is

$$f'_1 + \alpha f'_2 = A f_1 + B f_2 + C f_3$$

This relation can be of third order. At boundary point $i = 2$ the scheme for the first derivative in space is given by

$$\alpha f'_1 + f'_2 + \beta f'_3 = A f_1 + B f_2 + C f_3$$

This relation can be of fourth order. The solution coefficients for these two boundary schemes are determined in Gamet et al. [4].

Note that the nonuniform meshes used here is such that the gridpoints in the region where wave propagation, instability and transition processes take place are uniformly spaced. This implies that the finite difference scheme provides sixth-order accuracy in the uniform region. Moreover, the boundary between the uniform mesh and the coarsened mesh was chosen sufficiently far from the region where the instability processes govern the flow field. In addition, the vortex induced velocity field in the nonuniform mesh region is representative of a very weak uniform flow and may be assumed laminar.

### 2.2 Initial condition and numerical framework

For the study of bursting phenomenon in a vortex, the initial condition consists in a Lamb-Oseen vortex with two local core radius variations. This vortex model is characterized by a solid rotation in the core and constant circulation at infinity. The two-dimensional tangential velocity profile is defined in cylindrical coordinates by

$$v_\theta(r, t) = v_{\theta_{max}} \frac{r c}{r} \left(1 - e^{-\beta(r/r_c)^2}\right) = \frac{T_0}{2\pi r} \left(1 - e^{-\beta(r/r_c)^2}\right)$$

with $\alpha = 1.4$ and $\beta = 1.2564$. The pressure field is thus determined by

$$\frac{dP}{dr} = \frac{\rho v^2_\theta(r, t)}{r}$$
The vorticity distribution can be expressed as follows

\[
\omega_z(r,t) = \frac{\Gamma_0}{4\pi \nu t} e^{-\beta(r/rc)^2}, \quad rc = \sqrt{4\nu \beta t + rc_0^2}
\]  

(21)

The initial condition for the three dimensional simulations has been obtained by an extrusion of the two-dimensional vortex flow field in the axial direction. To generate a pressure wave, the same method of Moet et al. [12] is used, i.e., a variation of the core radius is imposed along the vortex axis as sketched in the figure 1. Two vortex regions are defined with constant radius, \(rc_1\) and \(rc_2\), which are connected by a section where the core radius evolves following a sinusoidal law. This vortex shape leads to a minimum underpressure variation in the vortex along the axis (Fig. 2). This simplified configuration allows to understand the physics of wave propagation along a vortex, which is a phenomenon observed in some experimental tests and during the reconnection process of Crow instability in the far-field. This perturbation is also similar to the propagation of axisymmetric perturbation as Kelvin waves with modes \(m = 0\). The main objective of the present study is to establish a criterion which determine the appearance of vortex bursting. A set of DNS simulations were performed changing the magnitude \(\varepsilon\) of the vortex core radius variation:

\[
\varepsilon = \left| \frac{rc_2 - rc_1}{rc_1} \right|
\]  

(22)

Note the circulation is set constant in longitudinal direction. The vortex region with the core radius \(rc_1\) is called the unperturbed region and this core radius is used as reference length scale. The Reynolds number is based on the circulation of the vortex: \(Re_F = \Gamma/\nu = 10^4\) and remains identical for all simulations.

The computational domain is defined in the transverse plane as \(L_x = L_y = 40rc_1\) and \(L_z = 81rc_1\) in the axial direction. Note the axial length of the computational domain has been chosen sufficiently large to isolate the dynamics of bursting. It consists of \(N_x \times N_y \times N_z = 145 \times 145 \times 541\) grid points. The mesh is regular in the region of interest for the dynamics of wave propagation and instability processes i.e. in \(L_{xp} = L_{yp} = 15rc_1\). In this region the regular mesh size is \(\Delta = 0.15rc_1\). The mesh is then stretched far away to minimize the boundaries effects.

Several kind of boundary conditions were tested for the transverse directions: the symmetric, the periodic and the non-reflecting boundary condition of Poinsot and Lele [15]. This latter has been chosen as it represents best a no confined flow and have less impacts on the integral quantities. Moreover, the choice of large transversal domain allows to minimize the effects of boundary conditions. Finally, periodic boundary conditions are employed in the longitudinal direction. This implies a modelling of a longitudinal vortex with a periodic signal characterised by a vortex core radius variation. Moreover, their employment allowed to perform Fast Fourier Transform in longitudinal direction without additional numerical treatment.
3 Computational results

This section presents the vortex bursting dynamics through the collision of two pressure waves propagating in opposite directions. The results are presented in non-dimensional form. The velocity is normalized by the initial maximum of azimuthal vortex velocity $v_{\theta_{\text{max}}}$ and length by the vortex core radius $l^* = l/rc_1$. The vortex region called una®ected region is denoted by subscript 1. The non-dimensional time is then computed by $t^* = t/T_{\text{rot}} = t/(2\pi rc_1/v_{\theta_{\text{max}}})$ ($T_{\text{rot}}$ being the vortex turnover period).

3.1 Global dynamics

The °ow con®guration with $\varepsilon = 1$ is chosen to illustrate the global dynamics of vortex bursting. The ®gures 3,4 represent the evolution of two selected vorticity isosurfaces. The global dynamics can be described by three phases. The ®rst one is characterised by the propagation in opposite directions of two annular vorticity structures along the vortex ($t^* = 0 - 4.46$). These structures are formed due to the axisymmetrical °ow property and the induced axial velocity, resulting from the pressure gradient in the vortex core (see [12] and TR112-2 [13]). For this con®guration $\varepsilon = 1$, the induced axial velocity is su®cient to render unstable the vortex °ow to three-dimensional disturbances. The second phase is the collision of the two waves leading to the vortex bursting ($t^* = 5.44 - 9.58$), with a sudden and drastic change of the vortex structure. Helical instability is developped at the time $t^* \approx 6.72$, which can be explained by the small perturbations resulting from the collision providing energy of the unstable modes. Afterwards, the waves get together leading to a larger vortex bursting $t^* \approx 9.58$. Finally, a third phase can be distinghised, where two others waves propagates in opposite directions of the collision station which lead again to a development of helical instability. Note that the initial symmetry of the °ow is remarkably conserved despite of the non-linear dynamics from the different phenomena.

The pressure pro®ls extracted at the vortex axis and plotted as function of the axial position show the waves propagation at a constant velocity (Fig. 5) before the collision, i.e $t^* \approx 0 - 5$. When these two waves collide the pressure minimum in the vortex core increases strongly. The propagation along the vortex axis of another waves resulting from the bursting is also observed. For the two flow con®gurations $\varepsilon = 0.0615$ and $\varepsilon = 1$, one can observed the same physical phenomenon of pressure waves propagation along the vortex core and their collision. However, the notable difference is the results of bursting. In the case of large wave magnitude, the vortex core is very disturbed at the location of collision, while in the other case the vortex core seems to return towards an undisturbed state. Results show that the vortex bursting and its consequences can be different following the magnitude of the waves.

An important consequence of the front wave propagation is to induce an axial velocity in the vortex core, which might be sufficient to destabilize the vortex flow as discussed in
3 Computational results

Here, for the configuration of large wave magnitude \( \varepsilon = 1 \), there is a first development of helical instability before and after the bursting. This can be explained looking the induced axial velocity \( W \) in the vortex core as function of the axial position. Indeed, this velocity is high enough to render the vortex flow unstable to three-dimensional disturbances. The swirl parameter \( q \sim 1.57 \star v_{\theta_{\text{max}}} / W_{\text{max}} \) is in the range of \( 0 < |q| < 1.5 \) of unstable flow as analysed notably by Mayer and Powell [9]. The first development of helical instability is followed by the vortex bursting and results in complex interactions. After the vortex bursting, new front waves propagate along the vortex in opposite directions with sufficient axial velocity to destabilize a second time the vortex flow, leading to a development of helical instabilities.

The next section deals with the quantification of this vortex dynamics, looking the circulation profiles, the enstrophy time evolution and the generation of small structures, in order to establish a status of this phenomenon.

3.2 Quantitative analysis

Vortex structure at the station of collision

This section presents the vortex structure analysis through the circulation profile in radial direction. The profile is extracted from the station \( z^* = Lz/2 = 40.5 \) corresponding to the station of collision, as the two waves propagate at the same velocity. It is defined by

\[
\Gamma(r, \theta) = \int \int \omega_z r dr d\theta
\]

(23)

The axial vorticity component is firstly interpolated into a polar grid, then integrated in the azimuthal direction to obtain a circulation profile depending only on the radial distance from the vortex center. The figure 7 shows the different circulation profiles at different times and for several flow configurations (characterised by the initial wave magnitude \( \varepsilon \)). It can be observed that the vortex structure is more disturbed increasing the wave magnitudes. These results show the sudden increase of the vortex core when the two waves intersect. Afterwards, \( t^* > 5.44 \), the circulation profile tends towards its initial shape. It can be explained by a dissipation process of the small scales structures at this station.

Evolution of enstrophy

Several calculations were performed with various wave magnitude \( \varepsilon \). This parameter governs the wave propagation velocity and one can expect that the vortex bursting will be more pronounced with higher waves speed. All simulations were done at the same Reynolds number \( Re_\Gamma = 10^4 \), based on the vortex circulation \( \Gamma \). The enstrophy is used for the first analysis. It is defined by

\[
Z = \frac{1}{V} \iiint_D \omega^2 dV
\]

(24)
where $D$ is the computational domain and $V$ its volume.

The figure 8 presents the time evolution of the enstrophy as function of the different waves magnitude $\varepsilon$ considered here. One can observed when the two waves collide ($t^* \sim 5.5 - 6.5$), that there is a sudden rise of enstrophy, which depends clearly to the waves magnitude. One obtained a peak of enstrophy only for the cases where $\varepsilon > 0.4$. Thus, it seems that the bursting associated with important consequences, occurs merely for large waves magnitude. Moreover, the end of simulation ($t^* > 10$) is marked by a second rise of enstrophy for the large wave magnitude configurations, which can be explained by the development of a helical instability.

This results shows also that the bursting takes place at different times, which is explained by the different waves speed. Indeed, increasing the waves magnitude results in higher waves propagation velocity. In order to establish a status of the vortex bursting phenomenon, we have plotted the enstrophy value as function of the wave front magnitude (Fig. 9) at the moment $t_c$ where the two waves cover. This time corresponds to the time for one front wave to travel along the vortex on the distance of $d^* = 35.5$ at the velocity propagation $v_p$. This latter depends to the wave front magnitude, but is very close to the maximum of the azimuthal velocity ($v_{\theta_{max}}$). Thus, $v_p$ is chosen such that $v_p = v_{\theta_{max}}$ to define $t_c$ independent to the flow configuration as possible ($t_c = 5.65T_{rot}$). This result shows that the collision of two waves propagating in opposite directions can be characterised by two states: one where the waves intersect but without leading to the vortex bursting ($\varepsilon < 0.4 - 0.5$) contrary to the second one where it occurs ($\varepsilon > 0.4 - 0.5$). The transition between the two states seems to be continuous.

Note:
Two others definitions of collision time $t_c$ can be used:
- The moment where the enstrophy reaches the first maximum for the cases where there is a rise of enstrophy, and for the other cases the same definition as before.
- Same definition as before but with the respective wave propagation velocity.

The behaviour of enstrophy extracted at this two moments, shows similar results, notably the change close to the value $\varepsilon \sim 0.4 - 0.5$.

### Generation of small structures

This section presents a spectral analysis of the kinetic energy in longitudinal direction in order to quantify the generation of small scales structures caused by the collision of the two waves.

To that end, the kinetic energy is calculated in spectral space. The velocity is decomposed in a Fourier serie:

$$u(x, y, z, t) = \sum_{n=0}^{N_z} \hat{u}(x, y, k_z, t)e^{ik_zz} \quad with \quad k_z = n\frac{2\pi}{L_z}$$  \hspace{1cm} (25)

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with $N_z = nz - 1$ the number of grid points in the longitudinal direction ($nz = 541$) and $L_z = 81$ the domain length. The Fourier transform is:

$$\hat{u}(x, y, k_z, t) = \frac{1}{L_z} \int_0^{L_z} u(x, y, z, t) e^{-ik_z z} dz. \quad (26)$$

The modal kinetic energy for each wavenumber $k_z$ in the longitudinal direction is defined by

$$E_{k_z}(t) = \frac{L_z}{2} \iint \hat{u} \hat{u}^* dxdy \quad (27)$$

where $\hat{u}^*$ is the complex conjugate of $\hat{u}$. Note that the integrals in the transverse directions are limited by the finite domain length ($L_x, L_y$).

The figure 10 presents the mono-dimensional spectra at different times for the case $\varepsilon = 1$. In this case the vortex bursting occurs and when the two waves collide $t^* \sim 4.46 - 5.44$, there is a continuous rise of the energy of the small scales structures, following by a dissipation $t^* \sim 6.72 - 9.58$. One can remarked a second increase at the time $t^* = 11.14$, which can be explained by the development of a helical instability in the vortex as discussed and described in section 3.1. These results confirm the first description of the vortex bursting dynamics: generation of small scales structures at the moment of collision, following by their dissipation, and eventually a second event (helical instability).

In order to establish a measure of the vortex bursting, it is proposed a quantification of the small scales structures generation. One defines a kinetic energy of small scales structures by

$$E_{sce}(t) = \sum_{k=k_c}^{NZ} E_k(t) \quad (28)$$

where $k_c$ is an arbitrary wavenumber. Here, $k_c = 81$ corresponding to the wavelength $\lambda/rc_1 = 1$. The kinetic energy of small scales structures is calculated at the time $t^*_c = 5.65$ and plotted in the figure 11.

For the cases of a wave magnitude $\varepsilon < 0.5$, the generation of small scales structures is very slight with respect to the other ones. The vortex bursting studied through the collision of two waves propagating in opposite directions, induces important consequences especially for large wave magnitudes $\varepsilon \geq 0.5$. 


4 Conclusion

The present Technical Report presented DNS simulations of the collision of two pressure waves propagating in opposite directions along a vortex. The waves are generated by a vortex core radius variation at its edges, which travel approximatively at a constant speed. This velocity depends mainly to the vortex properties in the unperturbed region (maximum of azimuthal velocity).

The global dynamics is governed firstly by the propagation of the two waves along the vortex, following by their collision or intersection, and then a dissipation and eventually a development of a helical instability. The vortex bursting leads to a drastic and sudden change of the vortex structure, especially in case of large wave magnitudes. Indeed, it was observed that for the waves magnitude lower than 40% of the vortex core radius, the collision of the two waves leads to small interactions and impacts slightly the vortex. The quantitative analysis, looking the evolution of enstrophy and the generation of small scales structures, showed that the vortex bursting leads to important changes for the cases of a wave magnitude higher of $40 - 50\%$ than the vortex core radius.
5 Figures

Fig. 1. Schematic sketch of the vortex radius variation along the axis. \( r_c_1 \) and \( r_c_2 \) are the vortex core radius.

Fig. 2. Axial pressure distribution in the vortex core, for different front wave magnitudes \( \varepsilon \).
Fig. 3. Illustration of vortex bursting by the evolution of two selected isosurfaces of vorticity magnitude. Time is normalized by the turnover period of the vortex $t^* = t/T$ with $T = 2\pi r c_1/\nu_{\text{max}}$. Configuration $\varepsilon = |r c_2 - r c_1|/r c_1 = 1$. 
Fig. 4. Illustration of vortex bursting by the evolution of two selected isosurfaces of vorticity magnitude. Time is normalized by the turnover period of the vortex $t^* = t/T$ with $T = 2\pi r_c/v_{\text{max}}$. Configuration $\varepsilon = |r_{c2} - r_{c1}|/r_{c1} = 1$. 

5 Figures
a) case $\varepsilon = 1$

b) case $\varepsilon = 0.0625$

**Fig. 5.** Time evolution of the pressure along the vortex core.
Fig. 6. Time evolution in the two particular axial station $z$ for the case of a large wave magnitude $\varepsilon = 1$. 

a) Axial velocity in the vortex core

b) Swirl number calculated with the actual axial and azimuthal velocity values
Fig. 7. Circulation profile at different times, as function of the radial distance from the vortex center, and for different wave magnitudes $\varepsilon$. The profiles are normalized by the initial total circulation $\Gamma_0 = 2\pi \alpha r_c v_{\phi,\text{max}}$. 
Fig. 8. Time evolution of enstrophy for the different front wave magnitude $\varepsilon$, normalized by its initial value.

Fig. 9. Peak of enstrophy at the time of vortex bursting or collision $t_c$, as function of the different wave magnitudes $\varepsilon$. 
Fig. 10. Kinetic energy spectrum at different times for the configuration $\varepsilon = 1$.

Fig. 11. Kinetic energy of the small scales structures at the collision time $t_c$, as function of the wave magnitude $\varepsilon$. 