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**Linear stability analysis of the elliptic instability in vortices with axial flow**

Prepared by: S. Le Dizès (CNRS-IRPHE)  
L. Lacaze (CNRS-IRPHE)

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Linear stability analysis of the elliptical instability in vortices with axial flow

By Laurent LACAZE and Stéphane LE DIZÈS
Institute de Recherche sur les Phénomènes Hors Équilibre (IRPHE)
CNRS / Universités Aix-Marseille I & II
49 rue F. Joliot-Curie, B.P. 146, F-13384 Marseille Cedex 13, France

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Vortices generated by aircraft wings are known to interact with each other. Each vortex is in particular subjected to the strain field induced by the other surrounding vortices. This strain field is at the origin of what is now called the elliptic instability which develops in vortex cores. For vortices without axial flow, this instability is characterized by a sinuous deformation of the vortices with a wavelength proportional to the vortex core size. The purpose of the present work is to demonstrate that other unstable modes are expected when there is an axial flow in the vortex core. The theory is developed for a Batchelor vortex in a stationary strain field. This model describes the situation in the far wake where the flow is composed of two counter-rotating Batchelor vortices. Similar qualitative conclusions are expected for other types of vortices, or vortices in rotating strain fields. This report is a part of an article in preparation for the Journal of Fluid Mechanics.
1. Introduction

A vortex which is elliptically deformed by an external strain field is generically subjected to the so-called elliptic instability. This instability has been extensively studied in vortices without axial flow. The purpose of this work is to analyze the effect of an axial flow on its occurrence and determine the elliptic stability characteristics of a classical model of vortex with jet.

The elliptic instability is now recognized as an important phenomenon of vortex dynamics. It is believed to take place in various contexts ranging from three-dimensional transition in shear flows (Bayly et al. 1988) to vortex interactions (Leweke & Williamson 1998) and flows in elliptic containers (Eloy et al. 2003). We refer to the recent review by Kerswell (2002) for details and other references. The generic aspects of the elliptic instability were first identified by Pierrehumbert (1986) and Bayly (1986) who considered the local stability properties of an elliptic flow. Before these local analyzes, Moore & Saffman (1975) and Tsai & Widnall (1976) had identified an instability which develops in strained vortices. They performed the first global stability analysis of the elliptic instability and provided an instability mechanism in terms of normal mode resonance. Moore & Saffman (1975) showed for an arbitrary strained vortex without axial flow that two neutral normal modes (Kelvin waves) of the underlying axisymmetric vortex are coupled by the strain field if their characteristics satisfy a condition of resonance. They also provided, by an asymptotic analysis in the limit of small strain field, a formal expression for the growth rate of the resonant modes. This theory has been applied to various vortices without axial flow (Tsai & Widnall 1976; Eloy & Le Dizès 2001, 1999; Fabre & Jacquin 2004).

The effect of axial flow has been considered only recently in Lacaze et al. (2005) for the Rankine vortex with a constant axial flow in its core. They showed that axial flow modifies the characteristics of the most unstable resonant modes. However, the Rankine vortex is a crude approximation for a realistic vortex. In particular, we now know that some of its normal modes disappear when the vortex profile is changed into a smoothly varying profile (Sipp & Jacquin 2003; Fabre et al. 2004). The vortex we consider here is a classical model of vortex with axial flow. It is known to model correctly the structure of trailing vortices in the far-wake of airplanes (Batchelor 1964).

In the aeronautical context, the elliptic instability is expected to intervene in the dynamics of the multiple vortices generated by aircraft wings. Each vortex is in the strain field of surrounding vortices, and therefore subjected to the elliptic instability. In configurations without axial flow, the elliptic instability has been experimentally observed in both counter-rotating vortices (Leweke & Williamson 1998) and in co-rotating vortices (Meunier & Leweke 2005). It has been modeled using Moore & Saffman approach in Le Dizès & Laporte (2002). They demonstrated that this approach based on a single strained vortex provides very good estimates for the elliptic instability characteristics in vortex pairs. A similar comparison will be performed in future work (Technical Report 1.2.1-3): The theory constructed for a single strained vortex will be validated by numerical results obtained for a pair of counter-rotating Batchelor vortices.

Batchelor vortex has been the subject of numerous works. It is known to be unstable with respect to inviscid perturbations when the axial flow exceeds a critical value (see for instance Ash & Khorrami 1995). Here we are not interested in this instability. The axial flow will be varied below this critical value. Recently, Fabre et al. (2004) have discovered that Batchelor vortex also exhibits unstable modes for small axial flow if the Reynolds number is sufficiently large. These modes are purely viscous and localized in the vortex center. Their growth rate is \(O(Re^{-1/3})\). These peculiar modes will not be considered in the present work. We shall consider the resonant coupling of inviscid normal modes only.
For small axial flow, these normal modes are expected to be either neutral or damped by a critical layer singularity. The appearance of critical layers is a common feature of normal modes in vortices with continuous vorticity profiles but very few information is available in the literature. Sipp & Jacquin (2003) studied these singular modes for the Lamb-Oseen vortex. Le Dizès & Lacaze (2005) provided partial informations on these modes for the Batchelor vortex using an asymptotic approach. An important part of the present work will be the determination of these modes for the Batchelor vortex as axial flow is varied.

The report is organized as follows. In section 2, the characteristics of a strained Batchelor vortex are described. Some properties of the linear Kelvin modes of the Batchelor are also provided. In section 3, the mechanism of the elliptic instability is recalled. The characteristics of the instability modes are given. Instability diagrams are computed for several values of the Reynolds number and strain rate. The conclusion briefly summarizes the main results.

2. Problem formulation

2.1. Description of the base flow field

The Batchelor vortex is a self-similar solution of the Navier-Stokes equations. Its axial vorticity \( \omega_o \) and axial velocity \( W_o \) can be written in cylindrical coordinates as

\[
\omega_o = \frac{\Gamma}{\pi R^2} e^{-\left(\frac{r}{R}\right)^2},
\]

\[
W_o = \frac{\xi R^2}{R^2} e^{-\left(\frac{r}{R}\right)^2},
\]

where the radius \( R(t) \) evolves in time by viscous diffusion according to

\[
R(t) = 4\nu t + R_0.
\]

The circulation \( \Gamma \) and axial velocity \( \xi \) are constants which measure the strengths of the rotation and of the jet respectively.

In the following, the viscous diffusion of the radius \( R(t) \) will be neglected and we will assume \( R(t) = R_0 \). This hypothesis is common in aeronautical applications where the Reynolds number often exceeds \( 10^6 \).

Variables are non-dimensionalized by the radius \( R_0 \) and the angular velocity in the vortex center \( \Omega_0 = \Gamma/(2\pi R_0^2) \) such that the above expressions become:

\[
\omega_o = 2e^{-r^2},
\]

\[
W_o = W_0 e^{-r^2}.
\]

The flow is characterized by the Reynolds number \( Re = \Gamma/(2\pi \nu) \) and the axial strength \( W_0 = 2\pi R_0 \xi / \Gamma \). The parameter \( W_0 \) can be considered as the inverse of the Swirl number.

Here, we shall assume that \( W_0 < 0.6 \) such that the vortex is stable in a inviscid framework (see Ash & Khorrarami 1995).

When a vortex is subjected to an external stationary strain field generated either by another vortex or by boundaries, its streamlines are deformed elliptically. The way an equilibrium solution is obtained when the strain field is small has been analyzed in Jiménez et al. (1996) and Le Dizès (2000a) for a single strained vortex without axial flow, and in Sipp et al. (2000) for a system of two-counter rotating vortices without axial flow. The presence of an axial flow does not modify the two-dimensional equilibrium solution because the dynamics of the axial velocity is decoupled from the other components of
the velocity. If we assume that the dimensionless external strain rate \( \varepsilon \) is small, a first order solution for the axial vorticity and for the axial velocity can be obtained as

\[
\begin{align*}
\omega_z &= \omega_0 - \varepsilon \frac{f(r) \omega_0}{2 \Omega_o r} \cos (2\theta) + \mathcal{O} (\varepsilon^2) , \\
U_z &= W_o - \varepsilon \frac{f(r) W'_o}{2 \Omega_o r} \cos (2\theta) + \mathcal{O} (\varepsilon^2).
\end{align*}
\]

(2.6) 

(2.7)

where \( \Omega_o = (1 - e^{-r^2})/r^2 \) is the angular velocity of the Batchelor vortex and \( f(r) \) satisfies

\[
\frac{d^2 f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr} - \left( \frac{4}{r^2} + \frac{3\Omega'_o}{r \Omega_o} + \frac{\Omega''_o}{\Omega_o} \right) f(r) = 0 .
\]

(2.8)

The function \( f(r) \) characterizes the interaction of the strain field with the vortex. For large \( r \), it satisfies \( f(r) \sim r^2 \) such that it matches the external strain field. Equation (2.8) was also given by Jiménez et al. (1996) and Eloy & Le Dizès (1999) in the case without axial flow. In the configuration of a Rankine vortex with jet, an explicit expression for the function \( f \) can be derived, as shown in Lacaze et al. (2005).

As already mentioned in the introduction, the elliptic instability can be interpreted as a process of resonance between linear modes of the underlying vortex (here the Batchelor vortex) with the correction induced by the strain field. In the next section, some of the properties of the inviscid normal modes of the Batchelor vortex are described. The condition of resonance are considered in section 3.1.

2.2. Inviscid linear modes of the Batchelor vortex

The velocity field \( \mathbf{u}' \) and pressure field \( p' \) of linear normal mode perturbations are searched in the form

\[
(\mathbf{u}', p') = (\mathbf{u}_K(r), p_K(r)) e^{ikz + im\theta - i\omega t}
\]

(2.9)

where \( \omega \) is the temporal frequency and \( k \) and \( m \) are the axial and azimuthal wavenumbers respectively. This expression inserted in the linearized Euler equations leads to a second order differential equation for the pressure amplitude \( p_K \) (see Le Dizès 2004). The dispersion relation of the linear modes is obtained by enforcing on \( p_K \) adequate boundary conditions at the origin and at infinity.

For the Rankine vortex without axial flow, the dispersion relation has been known for a long time (see for instance Saffman 1992). The linear modes are in such a case the so-called Kelvin waves first described by Kelvin (1880). For such a vortex, the Kelvin modes form a basis for perturbations confined within the vortex core (Arendt et al. 1997), and hence all perturbations acting within the vortex core may be described in terms of Kelvin modes. In addition, for any fixed axial wavenumber, \( k \) and azimuthal wavenumber \( m \), the possible frequencies \( \omega \) are discretized and within a fixed interval. In the case of a Rankine vortex with constant axial flow in the core, it has been shown that the Kelvin modes are slightly modified due to the presence of the jet (Loiseleur et al. 1998). Except for small values of \( k \), the jet mostly acts as a Doppler frequency shift.

The effect of a continuous profile on the characteristic of the modes is more complex. It has been the subject of several recent works (Sipp & Jacquin 2003; Fabre et al. 2004; Le Dizès & Lacaze 2005). For a Lamb-Oseen vortex (Batchelor vortex without axial flow), Sipp & Jacquin (2003) showed that linear inviscid normal modes become singular when their angular frequency is in the range of the angular velocity of the vortex. In such cases, the linear mode possesses a critical layer singularity. This singularity can be smoothed by introducing viscous effects (Le Dizès 2004) but the mode is no longer neutral: it has a
damping rate which becomes large when the critical layer singularity is close to the vortex center. An important point is that the damping rate does not depend on the Reynolds number for large Reynolds numbers. Moreover, the eigenfrequency can be computed by integrating the non-viscous equation in the complex plane above (resp. below) the singularity if \( m > 0 \) (resp. if \( m < 0 \)). Fabre et al. (2004) have recently shown the good agreement of this procedure with the results obtained from a fully viscous calculation. In figure 1 is displayed the dispersion relation of the modes \( m = 1 \) obtained for the Lamb-Oseen vortex by the contour deformation procedure. The branches are identified by a label \( i \) (\( i \) starts from 1 for the first retrograde mode). The frequency range where the eigenmodes possess a critical layer singularity has been indicated in figure 1(a). The damping rate of the critical layer modes is plotted in figure 1(b). It can be seen that the damping increases as the (real part of the) frequency of the mode increases in agreement with the displacement of the critical layer singularity toward the vortex center.

As long as axial flow is small, the picture of the temporal branches is not strongly modified (see also Le Dizès & Lacaze 2005). In figure 2 are presented the frequencies of the modes \( m = 1 \) for the Batchelor vortex with \( W_0 = 0.1 \). We recover the same structure of the branches as for the Lamb-Oseen vortex but a few differences can also be pointed out. The most important one is the modification of the frequency range of critical layer modes. In presence of axial flow, the frequency range of critical layer modes depend on the axial wavenumber. Axial flow has another important effect: it breaks the symmetry of the dispersion relation between the positive and negative azimuthal wavenumbers. We shall see in the next section that these two differences allow new linear mode resonance.

3. Elliptic instability characteristics

In this section, the main steps of the theoretical analysis leading to the characterization of the elliptic instability in a Batchelor vortex are provided. The effect of axial flow on the resonant modes is first quantified. Then, its influence on the instability growth rate is computed and a complete instability diagram is obtained as a function of the axial flow parameter \( W_0 \).

3.1. Characteristics of the principal modes

The theory of the elliptic instability is based on an asymptotic analysis with respect to the strain rate \( \varepsilon \), which is assumed small. The basic idea is presented in the work of Moore & Saffman (1975) and Tsai & Widnall (1976). For small \( \varepsilon \), the mechanism of the elliptic instability can be understood as a phenomenon of resonance: the vortex which is axisymmetric at leading order possesses neutral (or almost neutral) normal modes that can be resonantly coupled with the \( O(\varepsilon) \) correction induced by the external strain field. Upon remarking that this correction can be interpreted as a stationary (\( \omega = 0 \)), axially homogeneous \((k = 0)\) wave of azimuthal wavenumber \( m = \pm 2 \) [see expression (2.7)], the condition of resonance of two normal modes 1 and 2 with this correction is easily written as:

\[
\omega_2 = \omega_1 \quad k_2 = k_1 \quad m_2 = m_1 \pm 2 .
\]

The above condition is satisfied by numerous couples of normal modes. However, previous works on the Lamb-Oseen vortex (Eloy & Le Dizès 1999) and on the Rankine vortex (Eloy & Le Dizès 2001; Lacaze et al. 2005) have demonstrated that resonant configurations [satisfying (3.1)] corresponding to branches with the same label are in general the most unstable. We then first focus on these resonant configurations, which are called “principal modes” (Eloy & Le Dizès 2001).
Contrarily to the Rankine vortex (Lacaze et al. 2005), some inviscid normal modes are now damped by a critical layer singularity and thus they should not a priori be involved in any resonance. But, if the damping is small, the growth induces by the coupling could
still be larger. For this reason, we have chosen to monitor also the critical layer modes and allow the resonance of these modes as long as their damping rate is $O(\varepsilon)$.

In figures 3(a,b) are shown the wavenumber and the frequency of the first principal modes of azimuthal wavenumbers $(m_1, m_2) = (-1,1)$. For $W_0 = 0$, one can check that the results of Eloy & Le Dizès (1999) for the Lamb-Oseen vortex are recovered. In that case, all the principal modes $(-1,1)$ are stationary ($\omega = 0$). When $W_0$ is non-zero,
the frequency of the principal modes $(-1,1)$ is not zero anymore. This is due to the symmetry breaking mentioned above between the modes $m = 1$ and $m = -1$. One can also note that the different curves stop for a finite value of $W_0$: this is due to the strong critical layer damping of one of the two resonant modes (the curve has been stopped for $\Im m(\omega) < -0.04$). In figure 3(c) is displayed the dependence of this damping rate with respect to $W_0$. The other resonant mode $(m = -1)$ remains always neutral.

In figures 4(a,b) are shown the characteristics of the principal modes $(-2,0)$. It is worth mentioning that no resonance exist between such azimuthal modes without axial flow. This is due to the strong critical layer damping of the mode $m = -2$ for $W_0 = 0$. The variation of the critical layer damping rate is displayed in figure 4(c). We clearly see on this plot that axial flow can sufficiently modify the spectrum, especially the frequency range where critical layers are present such that new resonant modes become possible. We remind that in the case without axial flow, only the resonance $(m_1, m_2) = (-1,1)$ could arise. As the axial parameter is increased, the resonance $(-1,1)$ progressively disappears and is replaced by other resonances: $(-2,0)$, then $(-3,-1), (-4,-2)$, as it will be seen below. Each principal mode exists in finite intervals of $W_0$ in which the two resonant modes do not possess critical layers or in which one of the mode is neutral and the other one is only slightly damped by a critical layer. Only very few principal modes are possible. Note in particular that there are no principal modes $(m,m+2)$ with $m \geq 0$ for positive
axial flow. In the frequency range where branch crossing could have been possible, one of the two modes is indeed always strongly damped by a critical layer singularity.

In figures 3(a) and 4(a) has also been indicated by a small black disk on each curve the wavenumber of the principal mode for which

\[ \omega - kW_0 = \frac{m_1 + m_2}{2}. \]  

(3.2)

This condition corresponds to the condition of perfect resonance mentioned in Eloy & Le Dizès (2001) and Lacaze et al. (2005). As shown in Waleffe (1990) and Le Dizès (2000b), when this condition is satisfied, the resonant modes can be expressed near the vortex center as most unstable local plane waves for which a local estimate of the growth rate is \((9/16)\epsilon_0\) (\(\epsilon_0\) being the strain rate in the vortex center). Eloy & Le Dizès (2001) and Lacaze et al. (2005) have also shown that the instability is maximized when this condition is satisfied and that the local growth rate is a fairly good estimate for the elliptic instability growth rate in the inviscid limit. We shall see below that this condition also selects fairly well the most unstable configurations in the present case.

3.2. Growth rate of the instability

The growth rate of the resonant Kelvin modes can be computed by a multi-scale analysis, as shown in Moore & Saffman (1975). The velocity-pressure perturbation \( \mathbf{U}' = (u', p') \)
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is searched as a sum of two resonant modes of amplitude $A_1$ and $A_2$

$$U = A_1(\varepsilon t)U_{K_1}(r)e^{ik_1r+i\omega_1t} + A_2(\varepsilon t)U_{K_2}(r)e^{ik_2r+i\omega_2t},$$

where the wavenumbers and frequency of the two modes satisfy (3.1). From the equations at the order $\varepsilon$, two equations for $A_1$ and $A_2$ are obtained as orthogonality conditions. These equations possess solutions of the form

$$A_1(\varepsilon t) = B_1e^{\sigma \varepsilon t}; \quad A_2(\varepsilon t) = B_2e^{\sigma \varepsilon t},$$

which defines the normalized instability growth rate $\sigma$. The growth rate $\sigma$ is found to satisfy a relation which reads (see Elay & Le Dizès 1999, for details)

$$\left(\sigma J_{11} + ik\epsilon Q_{11} - \frac{1}{\varepsilon \text{Re}}\mathcal{L}_{11} - \frac{\text{Im} \omega}{\varepsilon} J_{11}\right) \times \left(\sigma J_{22} + ik\epsilon Q_{22} - \frac{1}{\varepsilon \text{Re}}\mathcal{L}_{22}\right) = \mathcal{N}_{12} \mathcal{N}_{21},$$

where

$$\mathcal{X}_{ij} = \langle U^A_{K_i}, \chi U_{K_j} \rangle,$$

and $U^A_{K_i}$ is the velocity-pressure amplitude of the adjoint Kelvin mode, solution of the adjoint operator obtained with the scalar product:

$$\langle X, Y \rangle = \int_0^\infty (\tilde{X} \cdot \tilde{Y}) r dr.$$

The term $J_{ij}$ in (3.5) represents the energy of the mode $i$, $\mathcal{N}_{ij}$ measures the coupling between the two modes and the strain field. The term $Q_{ii}$ permits to take into account a small wavenumber detuning with respect to exact resonance ($k_1 = k_2 = k_0 + \varepsilon k_2 + \mathcal{O}(\varepsilon^2)$ where $k_0$ corresponds to the value plotted in figure 3(a) or 4(a)). The term $\frac{\text{Im} \omega}{\varepsilon} J_{11}/\varepsilon$ is the damping term due to the critical layer (for which we implicitly assume $\text{Im} \omega = \mathcal{O}(\varepsilon)$). The viscous damping of each mode, given by the term $\frac{1}{\varepsilon \text{Re}}\mathcal{L}_{ii}$, has also been included in expression (3.5) in order to cut off small wavelengths. It is important to mention that including viscous effects on the perturbation is not in contradiction with our assumption of inviscid base flow. The viscous damping of large wavenumber Kelvin modes is indeed proportional to $k^2/\text{Re}$, and thus on a faster scale than the viscous time scale of the base flow if $k$ is large. For instance, if $k = \mathcal{O}(1/\varepsilon^{1/4})$ and $\text{Re} = \mathcal{O}(1/\varepsilon^{3/2})$, the viscous time scale of the perturbation is $\mathcal{O}(1/\varepsilon)$, whereas the viscous diffusion time scale of the base flow is $\mathcal{O}(\varepsilon^{-3/2})$.

If no damping term is taken into account, we obtain a simple inviscid estimate

$$\sigma_i = \sqrt{\frac{\mathcal{N}_{12} \mathcal{N}_{21}}{J_{11} J_{22}}}.$$

This expression is plotted for the first three principal modes $(-1, 1, i)$ and $(-2, 0, i)$ in figures 5 and 6 respectively. In these plots is also indicated by a small black disk the growth rate for the parameter satisfying the condition (3.2) of perfect resonance as indicated in figures 3(a) and 4(a). For the principal modes $(-1, 1)$, the condition of perfect resonance is obtained for $W_0 = 0$. It exactly corresponds to the axial flow value which maximizes the instability. One can also note that the maximum growth rate is also well-predicted by the local maximum growth rate estimate $(9/16)e_{\sigma}/\varepsilon$.

For the principal modes $(-2, 0)$, the perfect resonance growth rate is also close to the maximum growth rate and to the local maximum growth rate estimate. As for the case of the Rankine vortex (Lacaze et al. 2005), the condition of perfect resonance thus provides a good prediction of the parameters which maximize the instability. The inviscid
growth rate plotted in figures 5 and 6 does not take into account the “inviscid” damping associated with the critical layer. When this damping term is considered, the inviscid growth rate becomes dependant on the value of $\varepsilon$. This dependancy is illustrated for the principal mode $(-2,0,2)$ for two values of $\varepsilon$ in figure 7. One can check that the smallest the value of $\varepsilon$ the strongest the effect of the critical layer damping.

When the axial wavenumber detuning is considered, the growth rate of each principal mode becomes peaked in the $(W_0, k)$ plane near the parameter of perfect resonance. Typically, one obtains growth rate contours of the form illustrated in figure 8. The curve $k = k_0(W_0)$ as plotted in figure 4(a) for the principal modes $(-2,0)$ is displayed as a solid line. It is worth noting that the growth rate contours are not symmetric with respect to this curve. This is due to the critical layer damping which is more important on one side of this curve. The leading order variation of the (normalized) critical layer damping with respect to $k_0$ is given for the mode 1 by

$$\sigma_{CL}(k_0) = \Im \left( \frac{\Omega_{11}}{\Omega_{11}} \right) k_0 - \frac{\Im(\omega)}{\varepsilon}.$$  \hspace{1cm} (3.7)

The dashed line in figure 8 indicates the axial wavenumber $k = k_0 + \varepsilon k_0$ for which the critical layer damping approximated by (3.7) vanishes. Above this curve, expression (3.7) provides non-physical positive values for $\sigma_{CL}$ which have been replaced by zero in the general equation (3.5) for the growth rate. In figure 8, the thick solid line which sharply
cuts the growth rate contour delimits the region of existence of the principal mode. On this line, the branches $\omega(k)$ of the two Kelvin modes forming the principal mode are tangent with each other at the resonant point $k_0$. On the left of this line, the two branches $\omega(k)$ do not cross anymore and the principal mode does not exist. The growth rate expression (3.5) is not expected to apply near this line as higher order terms should be included to capture this topological change.

Contour plots as the one shown in figure 8 can be computed for each principal mode. When these plots are superimposed in the $(W_0, k)$ plane, we obtain a multitude of insta-
Figure 8. Illustration of the growth rate contours in the \((W_0, k)\) plane of a principal mode for fixed Reynolds number and \(\varepsilon\). Here the principal mode \((-4, -2, 1)\) for \(Re = 20000\) and \(\varepsilon = 0.01\). The curve of exact resonance \(k = k_0(W_0)\) is indicated in solid line. The dashed line limits the region where the critical layer damping is present in equation (3.5). The vertical cut limits the region of existence of the principal mode.

Figure 9. Instability area of the principal coupling modes in a plane \((W_0, k)\). Colors correspond to the intensity of the growth rate (from blue to red; from minimum to maximum) for \(Re = 20000\) and \(\varepsilon = 0.01\).

...bility “islands” associated with each principal mode. In figures 9 to 12, such instability diagrams are given for four couples of parameters \((Re, \varepsilon)\). Each “island” corresponds to a principal mode. One immediately sees that numerous instability modes are possible, each in a different region of the parameter space. In particular, the principal modes \((-1, 1, i)\) are seen no longer to be the only possible unstable modes in presence of axial flow. The other principal modes \((-2, 0, i), (-3, -1, i)\), etc, characterized by a more complex spatial structure become possible unstable modes as \(W_0\) is progressively increased.
Figure 10. Same as figure 9 for \( Re = 20,000 \) and \( \varepsilon = 0.015 \).

By comparing the figures 9, 10, 11 and 12 with each other, one immediately sees that the different instability regions spread as either \( \varepsilon \) or \( Re \) is increased, as expected. The variation with respect to the Reynolds number is also visible in figures 13 and 14. In these figures, the maximum growth rate contours and the characteristics of the most unstable principal modes are displayed in the parameter plane \((W_0, \varepsilon Re)\) for two values of \( \varepsilon \).

4. Conclusion

In this paper, we have analysed the stability of a strained Batchelor vortex with respect to the elliptic instability. Only small axial flow parameters have been considered for which the Batchelor vortex is stable with respect to inviscid perturbations. We have shown that axial flow modifies the characteristics of the elliptic instability. Without axial flow, the elliptic instability mode is formed of two stationary symmetric Kelvin modes \( m = 1 \) and \( m = -1 \). Axial flow breaks the symmetry between the \( m = 1 \) and \( m = -1 \) Kelvin modes such that the elliptic instability is no longer a sinusoidal stationary deformation in presence of a small axial flow. For larger axial flow, the resonance between Kelvin modes \( m = 1 \) and \( m = -1 \) disappears because one of the two modes becomes strongly damped due to a critical layer singularity. However, another resonance between \( m = 0 \) and \( m = -2 \) becomes possible leading to a new instability mode. As the axial flow is progressively increased, this resonance is replaced by another between modes \( m = -1 \) and \( m = -3 \) and so on. A complete instability diagram has been obtained for small and large Reynolds numbers demonstrating that the main characteristics are not strongly affected by the Reynolds number.

The nonlinear dynamics of the new instability modes is an important open issue. In particular, it will be important now to determine whether they can modify the dynamics of the vortex and enhance dissipation. The strain field being considered stationary, the
present configuration only applies to counter-rotating vortex pairs, that is to the far wake. In the far wake, axial flow is expected to reach in certain cases more than 10% the maximal azimuthal velocity: the new instability mode \((-2, 0)\) could therefore be present. It would be interesting to determine whether it can affect the development of the Crow instability.

In the near wake, axial flow is expected to be more important but the strain field acting on a given vortex is usually rotating. In that case, the theory is slightly different as the rotation of the strain field has to be taken into account in the conditions of resonance (see Le Dizès & Laporte 2002). However, this rotation is always small so the main results of the present analysis are not expected to be modified: the sinuous instability mode \((-1, 1)\) is expected to be superseded by other resonant modes \((-2, 0), (-3, -1)\), etc as axial flow increases. How these new modes modify the merging process of co-rotating vortices is among the questions which are now important to address.

REFERENCES


Figure 12. Same as figure 9 for $Re = 10^6$ and $\varepsilon = 0.01$.


Figure 13. Maximum instability growth rate (top) and associated most unstable principal mode (bottom) in the $(W_0, \varepsilon Re)$ plane for $\varepsilon = 0.01$.

Figure 14. Same as figure 13 for $\varepsilon = 0.1$.

