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Experiments on the elliptic instability in vortex pairs with axial flow

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Experiments on the elliptic instability in vortex pairs with axial core flow

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(19 September 2008)

In this paper, we present results from an experimental study of the dynamics of pairs of vortices, whose axial velocity in the cores differs from the one of the surrounding fluid. Co- and counter-rotating vortex pairs at moderate Reynolds numbers were generated in a water channel, at the tips of two rectangular wings. Measurements of the three-dimensional velocity field by means of stereoscopic Particle Image Velocimetry revealed a significant axial velocity deficit in their cores. For counter-rotating pairs, the long-wavelength Crow instability, involving symmetric wavy displacements of the vortices, could be clearly observed using dye visualisations. Measurements of the axial wavelength and growth rate of the unstable perturbation were found to be in good agreement with theoretical predictions, when the full experimentally measured velocity profile of the vortices, including axial flow, is taken into account. The dye visualisations further revealed the existence of a short-wavelength core instability. High-speed video recordings and Proper Orthogonal Decomposition of the visualisation time series allowed a precise characterisation of the instability mode, which involves perturbations of azimuthal wave number $m = 2$, as well as an axisymmetric wave ($m = 0$). These waves fulfill the resonance condition for the elliptic instability mechanism acting in strained vortical flows. A numerical three-dimensional stability analysis of the experimental vortex pair revealed the same unstable mode, and comparison of wavelengths and growth rates with the values obtained experimentally from dye visualisations, shows good agreement. Pairs of co-rotating vortices evolve in the form of a double helix in the water channel. For flow configurations that do not lead to merging of the two vortices over the length of the test section, the same type of short-wave perturbations could be observed. Quantitative measurements of wavelength and growth rate, and comparison with previous theoretical predictions, again identify them as an elliptic instability mode. The short-wave instability modes for vortices with axial flow are different from the one previously found in pairs without axial flow, which exhibits an azimuthal variation with wave number $m = 1$.

1. Introduction

The dynamics of a pair of vortices has been the object of a large number of studies in the last three decades, and with the development of new experimental techniques and increasing computer capabilities for numerical studies, the investigation of this flow has recently gained new impetus. The continued interest in this flow is, to a great extent, due to its relevance to the problem of aircraft trailing wakes, whose far field is primarily composed of a counter-rotating vortex pair. For large modern aircraft, these vortices can reach considerable strengths and represent a danger for following aircraft, especially smaller ones, due to the rolling moment and downwash they
induce. Therefore, a need exists to alleviate this danger by accelerating the decay of the wake. One approach is to try to take advantage of the natural “cooperative” instabilities that can occur in a vortex system. To reach this goal, solid knowledge about the characteristics of the instabilities is needed. Numerous international projects, listed, e.g., by Gerz, Holzäpfel & Darraçq (2002), were developed to investigate the problem. In addition to this practical aspect, the vortex pair also represents one of the simplest flow configurations for the detailed study of elementary vortex interactions, which may yield useful information for the understanding of the dynamics of more complex transitional or turbulent flows.

The first cooperative instability to be discovered in a vortex pair is a long-wavelength wavy instability, that occurs in a counter-rotating vortex pair: the so-called Crow instability. It can be observed in the sky behind aircraft flying at high altitude, when the wake vortices are visualised by condensation (see, e.g., the photographs shown in Scorer & Davenport 1970; Tombach 1973; Jacob 1995). Some laboratory observations can be found in Sarpkaya (1983); Liu (1992); Thomas & Auerbach (1994). The first theoretical analysis of this phenomenon was made by Crow (1970). He showed that the mutual interaction of the two vortices can lead to an amplification of displacement perturbations, whose axial wavelength is typically several times the initial vortex separation distance. The sinusoidal vortex displacements are symmetric with respect to the mid-plane between the two vortices, and they lie in planes inclined approximately at 45° with respect to the line joining the vortices. The origin of this instability is linked to the balance between the stabilising effect of self-induced rotation of the perturbations and the destabilising influence of the strain field that each vortex induces at the location of its neighbour. It was shown by Kelvin (1880), that a sinusoidal perturbation of a single vortex filament does not grow in time, but rotates around the vortex. In the presence of a second vortex, it can happen that the circumferential component of the velocity field induced by this vortex at the location of the first perturbed filament exactly cancels the self-induced rotation of the latter. The perturbation then remains in a stationary plane and is “pulled apart” by the radial component of the strain (which is also induced by the second vortex), leading to an exponential growth of its amplitude. A good description of this mechanism and its relation to Kelvin’s waves can be found in Widnall, Bliss & Tsai (1974).

Many subsequent studies, mostly theoretical and numerical, have illustrated and completed the work of Crow (1970). In particular, the effects of axial flow and arbitrary (axisymmetric) vorticity distributions in the vortices on the stability characteristics were analysed by Moore & Saffman (1973), and Klein et al. (1995). Extensions of Crow’s analysis towards systems of more than two vortices, representative of the near or extended near wake behind a transport aircraft (possibly with various flap configurations), have also been carried out. The volume of papers in Crouch & Jacquin (2005), which follows the reviews by Spalart (1998), Rossow (1999) and Gerz et al. (2002), comprises several of these studies. Although numerical simulations (Rennich & Lele 1997) seem to confirm the validity of Crow’s theory quite closely in the early stages of the instability, a precise experimental verification is still lacking. The few quantitative results from experiments (Sarpkaya 1983; Thomas & Auerbach 1994; Devenport et al. 1997), as well as observations from full-scale flight tests (Scorer & Davenport 1970), show qualitative agreement with theoretical predictions. However, no closer comparison was made. In this paper, we try to compare Crow’s inviscid theory with experimental results, concerning the wavelength and the growth rate of the long-wave instability in a pair of counter-rotating vortices with axial flow.

Short-wave instability, known as elliptic instability, is another cooperative instability that can be observed on a vortex subject to the influence of another one. Unlike Crow’s instability, whose wavelength is about eight times the vortex separation distance (Crow 1970), its typical wavelength scales with the radius of the vortex. It is called “elliptic instability” since it propagates on an elliptically deformed vortex. This instability can occur in counter-rotating as well as co-rotating configurations, contrary to Crow’s instability which is inhibited by the rotation of the
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strain (Jimenez 1975). Since the theory of Crow (1970) is based on the hypothesis of vortex filaments, which means that the vortices are considered as lines, Crow’s work only applies for long-wavelength instability. A short-wavelength mechanism such as the elliptic instability has to be explained using different considerations. This was done by Moore & Saffman (1975) who analysed linearly the stability of a finite-core vortex (with a small axial flow), deformed elliptically by the presence of a strain field, in an attempt to explain the unstable vortex rings visualised by Widnall & Sullivan (1973). The same analysis was carried out by Tsai & Widnall (1976). They confirmed Widnall’s proposed mechanism (Widnall et al. 1974) and concluded that the strain could resonate with two (neutral) Kelvin modes of the vortex, and lead to the exponential growth of a perturbation: the mechanism of the elliptic instability was discovered.

Later on, the numerical simulations of Pierrehumbert (1986) and the Floquet analysis of Bayly (1986) pulled the elliptic instability back into light. They showed that a two-dimensional inviscid flow with elliptical streamlines is unstable with respect to three-dimensional perturbations, in the short-wavelength limit. Landman & Saffman (1987) extended these results to viscous flows, arguing that the presence of viscosity imposes a minimum wavelength for the unstable perturbation. Waleffe (1990) linked the inertial wave approach to the Kelvin mode approach. He analysed the stability of a rotating flow in an elliptical container and showed that a combination of plane waves can lead to localised disturbances whose growth rates match with those given by the unbounded theory.

An overall picture of the elliptical instability was given in the review of Kerswell (2002). Concerning vortex pairs, the review mostly recalled theoretical and numerical work. Surprisingly, before the results of Leweke & Williamson (1998) who studied the elliptic instability in a pair of counter-rotating vortices, very few quantitative comparisons between predictions and experimental results had been performed. On vortex rings, impressively precise short-wave instability visualisations were presented by Krutzsch (1939). They show that the centre line of the vortex is displaced in the direction opposite to the direction of displacement of peripheral streamlines. This is typical of the elliptic instability in a pair of vortices without axial flow. A short-wave instability was also observed by Maxworthy (1972) and Widnall & Sullivan (1973). Malkus (1989) explored the nonlinear aspects of the instability in an elliptic flow bounded by an elliptical rotating cylinder. Using the same setup, Eloy, Le Gal & Le Dizès (2000) confirmed some elements of the theory presented by Waleffe (1990). Lacaze, Le Gal & Le Dizès (2004, 2005b) extended these results to the flow inside an elliptically deformed rotating sphere.

Thomas & Auerbach (1994) generated two parallel asymmetric vortices by rotating a sharp edge plate in a water tank. On top of the long-wave instability described by Crow (1970), they were able to visualise a short-wave instability, though the theoretical prediction of Widnall et al. (1974) did not seem to match the experimental results. A more rigorous and precise study carried out by Leweke & Williamson (1998) on the same vortex system, lead to an unambiguous conclusion: the experimental wavelength and growthrate of the short-wave instability observed are in agreement with the theoretical predictions of Tsai & Widnall (1976) and Waleffe (1990). This work was extended by Meunier & Leweke (2001, 2005) to co-rotating vortices. Numerical work of Billant, Brancher & Chomaz (1999) and Laporte & Corjon (2000) confirmed the conclusions of Leweke & Williamson (1998). So far, the theoretical predictions concerned uniform elliptical vortices (Waleffe 1990). To extend the results to more realistic configurations, Le Dizès & Laporte (2002) obtained an expression for the growthrate of the elliptic instability in a pair of co- and counter-rotating Gaussian vortices, depending on global parameters of the flow. They validated their results by Direct Normal Simulations (DNS) and Large-Eddy Simulations. They identified unstable bands corresponding to different axial wavenumber ranges. Each band is due to the resonance of the strain field (induced by the other vortex) with two Kelvin modes† for

† Though the denomination “Kelvin modes” (Kelvin 1880) strictly applies to the inviscid normal modes
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which the azimuthal wavenumber \( m \) and the frequency \( \omega \) are such that \( m = \pm 1 \) and \( \omega = 0 \). This is in agreement with the experimental investigations of Leweke & Williamson (1998) and Meunier & Leweke (2001), and the numerical work of Sipp & Jacquin (2003). Some studies (see, e.g., Fabre 2002; Fabre & Jacquin 2004) characterised the short-wave instability on vortices more representative of aircraft wakes.

All previous studies were performed on vortices without axial flow, except the early study of Moore & Saffman (1975). The first step to fill this gap was taken by Fabre, Cosus & Jacquin (2000), who discussed the convective and absolute aspect of the instability on a pair of Rankine vortices in a uniform axial flow field, subject to a weak strain. Lacaze, Birbaud & Le Dizès (2005a) analysed the stability of a Rankine vortex with an axial velocity core in a stationary strain. They discovered that, contrary to the case without axial flow, some Kelvin modes with high azimuthal symmetry orders and non-zero frequencies, could resonate, leading to the instability of the system. The same conclusions were drawn for a Gaussian vortex by Lacaze, Ryan & Le Dizès (2007). For parallel vortices, this corresponds to the counter-rotating case. Recently, these results were numerically extended to the co-rotating case by Roy, Schaeffer, Le Dizès & Thompson (2008b).

Experimental references concerning the elliptic instability in a vortex with axial flow are rare. Some visualisations of a short wave phenomenon in the wake of an aircraft can be found in Scorer & Davenport (1970), Jacob (1995) and Bristol et al. (2004). Similar observations were reported by Chen, Jacob & Savaş (1999) and Ortega, Bristol & Savaş (2003) who studied a four-vortex system generated by a flapped wing in a towing tank. The only quantitative data were presented by Devenport, Zsoldos & Vogel (1997) and Devenport, Vogel & Zsoldos (1999). They investigated counter- and co-rotating vortices generated by two symmetrical wings in a wind-tunnel. Some peaks in the velocity spectrum measured by hot-wire anemometry seem to be related to the elliptic instability.

In this paper, results are presented concerning experimental co- and counter-rotating vortex pairs with axial flow, that prove the presence of the elliptic instability. In particular, we analyse the spatio-temporal structure of a short-wavelength perturbation and relate this to the elliptic instability. More specifically, the axial wavelength, the azimuthal wavenumber and the growthrate of the unstable mode are obtained experimentally for a pair of counter-rotating vortices and compared with numerical results. First results of this study can be found in Roy et al. (2008a). The spatial characteristics of the unstable mode observed on a co-rotating pair are also given and compared to numerical predictions.

2. Experimental setup

2.1. Facility

The facility used for the experiments is a recirculating water channel with a free surface. It has a test section of dimensions 37 cm (width) \( \times \) 50 cm (height) \( \times \) 150 cm (length). The free-stream velocity \( U_\infty \) can be chosen in the range 5–100 cm/s. The turbulence intensities associated with the streamwise and transverse velocity components are approximately 1.5% and 0.6%, respectively. The bottom and side walls of the test section are made out of glass. An additional glass window downstream of the test section on the wall normal to the stream allows visual access to the flow inside the test section from five different directions. A schematic of the test section is shown in figure 1.
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2.2. Vortex generating wings

Two NACA0012 rectangular half-wings with a chord $c = 10$ cm made of polyvinyl chloride, were placed tip to tip in the free–stream of the water channel. They had a round tip with a varying tip diameter equal to the local thickness of the wing. The wings were mounted vertically at the upstream end of the test section. The first half-wing, with a 15 cm span, was fixed on a U-frame positioned along the side and bottom boundaries of the channel. It was possible to change the angle of attack of both wings. The second wing was fixed to a support outside the water, and traversed the free surface. The setup is illustrated in figure 2(a). A similar kind of setup was used by Devenport et al. (1997, 1999).

When a rectangular wing is placed in a uniform stream, it is now well known that a "wing-tip" vortex is generated (see, e.g., Mason & Marchman 1972; Baker, Barker, Bofah & Saffman 1974; Thompson 1975; Singh & Uberoi 1976; Katz & Bueno Galdo 1989; Green & Acosta 1991; Devenport, Rife, Liapis & Follin 1996). The sign of the circulation depends on the sign of the angle of attack of the wing. With one vortex being generated by each wing, one can generate a counter- or co-rotating vortex pair by imposing independently the angle of attack of both wings. Another advantage of this setup, compared to, e.g., a flapped wing, is the possibility to translate the top wing along its span axis. The spanwise position of the vortex is close to the wing-tip and mostly depends on the wing-tip shape (Hoerner 1965). Therefore, translating the top wing implies changing the separation distance $b$ between the two vortices. The three degrees of freedom of the setup were widely utilised in the exploration phase of the study, the aim being to find an unstable flow, exhibiting the elliptic instability. The origin $O$ of the frame of reference $\{O, x, y, z\}$ chosen for our study is the middle point of the line linking the wing tips. The $z$ axis points in the free–stream direction and the $y$ axis is parallel to the vertical direction, pointing up. The orthonormal frame is completed by the $x$ axis in the horizontal transverse direction.
A dye injection system allowed the visualisation of the vortices. A small pipe network was machined inside the U-frame and the wings to allow the injection of fluorescent dye (aqueous solution of fluorescein). The injection holes were located close to the wing tip. Different injection holes were available; depending on the configuration of the wings, the most appropriate ones were selected. The injection was ensured by a pressure difference. Special care was taken to make sure that this did not perturb the flow.

The dye injected in the flow was illuminated in volume by an argon ion laser (model Stabilite 2017 from Spectra Physics) coupled to an optical fibre with a spherical lens. The laser was oriented in the axis of the vortices through the downstream visualisation window presented in figure 1. Side view photos of the vortices were taken with a camera (Nikon D200) looking through the side walls of the channel test section. When a high acquisition frame rate was needed, a monochrome high-speed video camera (Vision Research Phantom V5) was used. Some visualisations of the vortices were taken in a plane normal to the free–stream. In that case, the optical fibre was coupled to a cylindrical lens, the resulting laser sheet being oriented from bottom to top. The D200 or Phantom camera was then looking through the downstream visualisation window.

All velocity measurements were made using stereoscopic particle image velocimetry (Stereo-PIV). This technique allows the measurement of three velocity components in a plane. Successful implementations of this method are presented, e.g., by Willert (1997), Alkislar, Krothapali & Lourenco (2003) and Carlier & Stanislas (2005). A Nd-YAG pulsed laser was positioned underneath the test section, at the desired distance from the wing, generating a 3 mm thick vertical light sheet (see figure 1). The flow was seeded with silver-coated particles (Dantec), whose size (100 µm diameter) was small compared to the characteristic scale of the vortex (diameter typically of the order of 1 cm). Pictures of the particles in the laser sheet were taken with two high-resolution digital cameras (Roper Redlake, 4000 pixels × 2672 pixels). Each camera was placed on one side of the test section (see figure 1), viewing the laser plane in a direction forming a 30° angle with the free–stream. The camera sensor and the lens were mounted in a Scheimpflug setup (Scheimpflug 1904). To conserve the orthogonality between the cameras’ line of sight and the air-liquid interface, water-filled glass prisms were fixed on the test section side walls (see figure 1). This reduces the distortion entailed by the non-orthogonal angle of sight. In addition, the small gaps between the prisms and the channel side walls were filled with water to alleviate the reflection of the light at the air-glass interfaces. An overview of the Stereo-PIV technique coupled with the use of prisms and Scheimpflug mounts can be found in Prasad & Jensen (1995) and Zang & Prasad (1997). The computing algorithm used to process the images is based on a
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2.5. Parameters of the flow

For each vortex, a Reynolds number $\Gamma/\nu$, based on the total circulation, is defined. Another parameter is the non-dimensional core size $a/b$, where $a$ is the vorticity radius (defined later) and $b$ the vortex separation distance. $a/b$ gives a measure of the interactions between the vortices. The axial flow parameter $W_0 = (U_0 - U_\infty)2\pi a/\Gamma$, where $U_0$ is the axial velocity at the centre of the vortex and $U_\infty$ is the free-stream velocity, compares the axial velocity defect on the vortex axis with a characteristic azimuthal velocity. We also introduce $a_w/a$, the ratio between the radial scale $a_w$ of the axial velocity profile and the core radius $a$.

3. Counter-rotating vortex pair

In this part, we focus on the three-dimensional instabilities of a pair of counter-rotating vortices. After an exploration of different parameter ranges, one configuration was chosen for a more detailed analysis. First, we describe the experimental base flow measured in the water channel with the Stereo-PIV technique. After a brief analysis of the long-wavelength Crow instability, we present visualisations of a short-wavelength instability mode of the vortices. In order to make the link between this observation and the elliptic instability theory, we estimate the growthrate, the wavelength, and the azimuthal wavenumber of the observed instability mode. Visualisations and Proper Orthogonal Decomposition (POD) techniques are used to extract the spatial structure and the temporal dependence of the instability mode. These data are compared with results of a numerical stability analysis of the experimental base flow.

3.1. Three-dimensional base flow

The wings were positioned in the counter-rotating configuration, in the middle of the water channel section. Both angles of attack were equal to $8.5^\circ$. The wing tips were 3 cm apart. The free-stream, approximately equal to 56 cm/s (see below). Two vortices were generated at the wing tips. The top vortex circulation was positive so that the vortex pair translated in the positive $y$ direction by mutual induction. In the following, indices 1 and 2 refer to the top and bottom vortices, respectively. Stereo-PIV measurements were performed in two planes normal to the axis of the water channel, located at $z/c = 5.6$ and $z/c = 9.4$. For each plane, 300 three-dimensional velocity fields were obtained using the procedure described in 2.4.

The mean axial vorticity ($\zeta$) and axial velocity ($U$) distributions at $z/c = 5.6$ are presented in figure 3. The axial velocity field presents a large deficit in the vicinity of the vortices. Residuals of the velocity defects behind the main parts of the wings can also be identified, to the left of the vortex cores. Driven by their respective vortices, the wake of the two half-wings is advected in the vicinity of the middle of the vortex pair. The axial velocity distribution in the vortex core is therefore strongly non-axisymmetric. This fact is reinforced by the presence of a small axial velocity excess located about 1 cm away from the top vortex, in the top-left direction (see figure 3). A small symmetrical decrease of the axial velocity can be observed close to the bottom vortex, which is difficult to explain. Though the axial velocity fields presented by Devenport et al. (1997), who used the same half-wing setup in a wind-tunnel, did not exhibit any velocity excess, a rapid increase of the axial velocity can be found in approximately the same position relatively to the vortex pair.

The stretching direction of the average axial velocity distribution in the vortex cores is approximately normal to the stretching direction of the average axial vorticity distribution, shown...
in figure 3). The elongation of the time-averaged vorticity distribution is mainly due to the motion of the vortices in time, in a preferred direction. To characterise this motion, the location of the vortices was determined for each of the 300 individual fields at $z/c = 5.6$ and $z/c = 9.4$. This was done by fitting the measured two-dimensional velocity field with the two-dimensional field of a pair of axisymmetric counter-rotating Gaussian vortices. The free parameters of this fit were, for each vortex, the total circulation, the Gaussian radius and the coordinates of the vortex centres in the $(O,x,y)$ plane. In order to characterise the vortex motion, the eigenvalues $\alpha_M^2$ and $\alpha_m^2 (\alpha_M > \alpha_m)$ of the covariance matrix of the vortex positions were computed. The eigenvector corresponding to $\alpha_M^2$ is the direction in which the variance of the vortex position projection is maximised. The normal direction minimises the variance $\alpha_m^2$. Physically, the direction corresponding to the $\alpha_M$ eigenvector is the preferred direction of motion of the vortex. We can illustrate this graphically by an ellipse centred on the average position of the vortex, of major and minor axes $a_M$ and $a_m$, oriented accordingly to the respective eigenvectors. The preferred direction of motion is characterised by the angle $\alpha$ formed by the line joining the average positions of the vortices and the preferred direction of motion of one vortex (see figure 4). For each vortex, the values obtained for $\alpha$, $a_M$ and $a_m$ are listed in table 1. Two physical mechanisms can explain the increase of $a_m$ and $a_M$ with $z$. The first one can be linked to the dynamics of a single vortex. The same trend was observed by Devenport et al. (1996) who used hot-wire anemometry to estimate the amplitude of the lateral motion of a single trailing vortex, the so-called “vortex meandering” (Roy & Leweke 2008). This phenomenon, is currently not fully understood; a probable explanation is related to transient growth of perturbations triggered by random external noise (Antkowiak & Brancher 2004; Fontane et al. 2008; Roy & Leweke 2008). The development of the Crow instability is the second factor leading to an increase of $a_M$ and $a_m$, as the amplitude of the long sinusoidal deformations of the vortices increases. This aspect of the flow is discussed in section 3.2. The mean separation distance $b$ between the vortices increases by 10% between $z/c = 5.6$ and $z/c = 9.4$. The same trend was observed by Devenport et al. (1996).

The transverse displacement of the vortices due to vortex meandering and Crow instability
has an effect on the fields presented in figure 3: the size of the vortex cores are overestimated, especially in the preferred direction of motion, and the axial velocity in the centre of the vortices is underestimated. It is possible to minimise this effect by taking advantage of the knowledge of the vortex positions before averaging the fields. Each field is translated so that the centre of one vortex centre lies in its previously computed mean position, before averaging the fields. The operation is then repeated for the second vortex.

For each vortex, the total circulation $\Gamma$ was estimated by integrating the velocity on a rectangular contour surrounding the vortex. The contour was chosen as large as possible to take into account as much vorticity as possible. One side of the contour was the centre line separating the two vortices. The values obtained show that the top vortex was stronger than the bottom vortex (see table 1). This difference in circulation entails a rotation of the pair around the vorticity centre. The point vortex model provides an estimation of the angular velocity of the vortex pair $(\Gamma_1 + \Gamma_2)/2\pi b^2 = 2.8^\circ/s$ (computed for $b = 4$ cm) which is coherent with the experimentally determined value $3.1^\circ/s$. The experimental translation velocity of the pair, 6.4 cm/s, is also in
good agreement with the translation velocity $\Gamma/2\pi b = 6.2 \text{ cm/s}$ of two point vortices of opposite circulation $\Gamma = 157 \text{ cm}^2/\text{s}$, with a separation distance $b = 4 \text{ cm}$.

The vorticity profiles were evaluated by computing the azimuthal average of the mean vorticity field obtained by the recentering method described above. It was fitted with a Gaussian function

$$
\zeta(r) = \frac{\Gamma}{\pi a^2} \exp \left(-\frac{r^2}{a^2}\right),
$$

(3.1)
to extract a vorticity radius $a$. As shown in figure 5 for the top vortex (index 1), the Gaussian function matches well the experimental data. A slight local underestimation of the vorticity is only observed around the radial position $r/a = 1.8$. Since the circulation was imposed during the fit, it only implies a small error on the spatial distribution of the vorticity but the overall vorticity is correctly represented.

For each vortex, $U_\infty$ is defined as the value of the axial velocity profile at $r/a = 3$. By subtracting $U_\infty$ from the axisymmetric axial velocity profile, we define the axial velocity defect profile $W$. $U_0$ is the axial velocity at the centre of the vortex. As the vorticity, $W$ is fitted to a Gaussian function (see relation (3.2) and figure 5), the axial velocity radius $a_w$ being the only free parameter.

$$
W(r) = U_0 \exp \left(-\frac{r^2}{a_w^2}\right).
$$

(3.2)
The match is excellent. The values of all vortex parameters are listed in table 1. We note an increase of the radius $a$ much faster than what viscous diffusion would suggest at this Reynolds number. At $z/c = 9.4$, the flow is highly turbulent, subject to vortex meandering and Crow instability. In these conditions, the determination of the vortex centre in individual PIV measurements is subject to errors and uncertainties, which lead to a spurious broadening of the average fields, despite the use of the recentering method.

### 3.2. Crow instability

In this section, the long-wavelength instability occurring in a counter-rotating vortex pair is described, since it is a major characteristic of the investigated flow. Theoretical elements will be recalled and compared with the experimental results.
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Figure 6. Visualisation of the Crow instability of a pair of counter-rotating vortices with axial core flow. The short-wavelength oscillation corresponding to the elliptic instability is also visible. The field view is approximately 7 cm \( \times \) 24 cm

As shown by the visualisation presented in figure 6, the vortices are subject to the long-wavelength instability. This phenomenon was first analysed by Crow (1970). He considered two parallel line vortices of equal and opposite circulations, whose dynamics are determined by Biot-Savart induction. The velocity induced at a given point on one of the vortices can be split up into two main parts: (1) the flow field induced by the other vortex, where the corresponding Biot-Savart integral can be calculated in a straightforward way, and (2) the self-induced motion, which must be calculated using an appropriate cut-off length to avoid the divergence of the Biot-Savart integral. An arbitrary perturbation of the initially straight and parallel vortex lines can be decomposed into a set of normal modes, which consist of plane sinusoidal displacements parameterised by their axial wavelength \( \lambda_c \), their orientation with respect to the line joining the initial vortices, and their symmetry: either symmetric with respect to the central plane of the pair, as exhibited by the mode in figures, or antisymmetric, whereby the perturbation of one vortex is shifted by half a wavelength in the axial direction compared to the corresponding symmetric mode.

The stability of these perturbations is governed by the interaction of three effects on a given vortex, which are all of the same order of magnitude. (1) The motion induced by the unperturbed other vortex. In the frame of reference moving with the vortex pair, the flow induced by one vortex at the location of the other unperturbed position is to first order a plane stagnation point flow, with maximum stretching in the 45\( ^\circ \) direction (Widnall et al. 1974). (2) The motion induced by the perturbations of the other vortex. To first order, this also yields a linear flow field in the vicinity of the other vortex centre, but with a more complicated angular dependence. (3) The self-induced motion, which, for the kind of perturbations considered here, consists of a pure rotation of the perturbation plane (Kelvin 1880) without growth of the amplitude, i.e., a third type of flow depending linearly on the spatial coordinates measured from the unperturbed given vortex position.

In the following, we concentrate on the symmetric perturbation modes of the vortex pair, since the anti-symmetric ones turn out to be stable. Following Crow (1970), an expression for the growth rate \( \sigma_c \) of the (symmetric) perturbations can be obtained, which depends on the wave number \( k_c = 2\pi/\lambda_c \) of the modes (\( \lambda_c \) is the axial wavelength), and on the core radius \( a \) of the vortices. In this analysis, the vortices are essentially treated as filaments without internal structure, which allows the use of Biot-Savart-type integrals to obtain the equations of motion for points on the vortices. The core radius enters in the evaluation of the self-induced velocity of each filament: it is related to the cut-off length to be used in the corresponding integral. The numerical relation between the two was established by considering a Rankine vortex model, i.e., vortices with constant vorticity inside a tube of radius \( a_c \) and zero vorticity outside (Crow 1970). (It will be recalled below, from Widnall et al. (1971) and others, that, for the long-wavelength dynamics, a vortex with arbitrary vorticity distribution can always be replaced by a Rankine vortex with an “equivalent” core size \( a_c \), which can be calculated).
The result for the non-dimensional growth rate \( \sigma^*_c = \sigma_c \cdot \left( \frac{2\pi b^2}{\Gamma} \right) \) is

\[
(\sigma^*_c)^2 = \left[ 1 - (k_c b)^2 K_0(k_c b) - k_c b K_1(k_c b) - \frac{\eta}{(a_e/b)^2} \right] \left[ 1 + k_c b K_1(k_c b) + \frac{\eta}{(a_e/b)^2} \right],
\]

(3.3)

where the \( K_i (i = 0, 1) \) are modified Bessel functions of the second kind and of order \( i \). \( \eta \) is the self-induced angular velocity of a sinusoidally perturbed vortex filament, whose plane is known to rotate around the axis given by the unperturbed straight vortex. \( \eta \) is non-dimensional, it is normalised using the angular velocity of the fluid at the centre of the vortex, which is equal to half the vorticity there.

The exact dispersion relation for all for sinusoidal displacement perturbations of azimuthal wave number \( m = 1 \) (often also called “bending wave”) of a Rankine vortex can be deduced from Kelvin’s (1880) original work. It is a solution of the following implicit equation:

\[
\frac{1}{\beta a_e} J_1(\beta a_e) + \frac{1}{k_c a_e} \frac{K'_1(k_c a_e)}{K_1(k_c a_e)} + \frac{\sqrt{\beta^2 + k_c^2}}{\beta^2} = 0,
\]

(3.4)

where \( J_1 \) is the Bessel function of the first kind, and \( K_1 \) the modified Bessel function of the second kind, both of order 1. The prime (’) denotes a derivative, and the parameter \( \beta \) is related to the rotation rate through

\[
\eta = 1 - \frac{2k_c}{\sqrt{\beta^2 + k_c^2}}.
\]

(3.5)

In practice, for a given \( k_c a_e \), (3.4) is solved numerically for \( \beta a_e \), and \( \eta(k_c a_e) \) is found by (3.5). Due to the oscillatory nature of \( J_1 \), equation (3.4) is verified for an infinity of values \( \beta a_e \) for each \( k_c a_e \). The pure displacement perturbations considered here are associated with the one value verifying \( 0 < \beta a_e < j_1 \), where \( j_1 \approx 3.8317 \) is the smallest root of \( J_1 \). The other values represent more complicated perturbation modes with internal core deformations.
Experiments on the elliptic instability in vortex pairs with axial core flow

The self-induced dynamics of vortices with arbitrary (axisymmetric) velocity profiles was analysed by Widnall et al. (1971), and subsequently by Moore & Saffman (1973), Leibovich et al. (1986), and Klein & Knio (1995), in the limit of long wavelengths. It can be deduced from this work that a vortex with azimuthal and axial velocity distributions \( v_\phi(r) \) and \( v_z(r) \) evolving on a characteristic radial scale \( a \), exhibits the same self-induced dynamics as an equivalent Rankine vortex, having the same circulation \( \Gamma \) and a core radius

\[
a_e = a \cdot \exp \left[ \frac{1}{4} - A + C \right].
\]  

(3.6)

A and C are integral parameters characterising the velocity profiles; they are given by

\[
A = \lim_{r \to \infty} \left[ \frac{4\pi^2}{\Gamma^2} \int_0^r \tau v_\phi^2(\tau) d\tau - \ln \frac{r}{a} \right],
\]  

(3.7)

\[
C = \frac{8\pi^2}{\Gamma^2} \int_0^\infty \tau v_z^2(\tau) d\tau.
\]  

(3.8)

Using the Gaussian functions modelling the top vortex profiles, we can theoretically determine \( A \) and \( C \) for the top vortex by means of (3.7) and (3.8). Numerically, we obtain \( A_1 \approx -0.0580 \) and \( C_1 \approx 0.0639 \). Therefore,

\[
a_e \approx 1.4575a.
\]  

(3.9)

Using (3.3)–(3.5), we can obtain a prediction for the growth rate of the long-wavelength instability in our pair of counter-rotating vortices.

The experimental wavelength \( \lambda_c \) was measured by taking the average of 200 measurements on visualisations obtained at \( z/c = 9.2 \). We find \( \lambda_c/b = 6.55 \), with an standard deviation of 0.6. An estimation of the experimental growth rate \( \sigma_c \), can be obtained by comparing the amplitude of \( a_M \) for the top vortex, between \( z/c = 5.6 \) and \( z/c = 9.4 \), with an estimated uncertainty of \( \pm 20\% \). Experimentally, we find

\[
\frac{\lambda_c}{b} = 6.55 \pm 0.60 \quad \text{(experimental)}.
\]  

(3.10)

Figure 7 compares the theoretical prediction with the experimental measurement. The agreement between the two results is very good. The theoretical maximum growth rate and corresponding wavelength, determined from (3.3), are:

\[
\frac{\lambda_c}{b} = 7.07 \quad \text{(theoretical)}.
\]  

\[
\sigma_c^* = 0.80 \quad \text{(theoretical)}.
\]  

(3.11)

3.3. Elliptic instability

3.3.1. Theoretical aspects

The theory of the elliptic instability is based on linear normal modes. Following this formalism, the perturbation mode velocity fields \( u' \) and pressure field \( p' \) are written as

\[
(u', p') = [u(r), p(r)] \exp(ikz + im\theta - i\omega t),
\]  

(3.12)

where \( k \) and \( m \) are the axial and azimuthal wavenumbers and \( \omega \) is the frequency. For the case of a Rankine vortex without axial flow, the linear normal modes solutions of the relevant dispersion relation \( D(k, m, n) = 0 \), are the so-called Kelvin modes. For each azimuthal wavenumber \( m \), there exists an infinity of branches in the \((k, \omega)\) plane, that are solutions of the dispersion relation. To
classify these branches, the label $n$ is commonly used. It corresponds to the number of zeros of the radial velocity $u_r$, in the interval $0 < r < a$. Arendt, Fritts & Andreassen (1997) showed that Kelvin modes form a basis for the perturbations localised in the vortex core. This means that any perturbation in the core of a Rankine vortex can be expressed by a combination Kelvin modes. Moore & Saffman (1975) and Tsai & Widnall (1976), who were not aware of this property, nevertheless used Kelvin modes to identify the mechanism of the elliptic instability. They studied a vortex with finite core size immersed in a weak strain field of second-order azimuthal symmetry. They showed that the strain field could resonate with two Kelvin modes of azimuthal wave numbers $m_1 = 1$ and $m_2 = -1$.

These results can be generalised for higher azimuthal wave numbers (Eloy & Le Dizès 2001). The general resonance condition coupling the strain with two Kelvin modes with $(k_1, m_1, \omega_1)$ and $(k_2, m_2, \omega_2)$, reads:

$$
\begin{pmatrix}
k_1 \\
m_1 \\
\omega_1
\end{pmatrix} = \begin{pmatrix}
k_2 \\
m_2 \\
\omega_2
\end{pmatrix} \begin{pmatrix}0 \\ 2 \\ 0 \end{pmatrix}.
$$

(3.13)

Two Kelvin waves can therefore resonate with the strain if their axial wavenumber and frequency are equal and their azimuthal wavenumbers differ by 2. Eloy & Le Dizès (2001) showed that for the case $m_1 = -1, m_2 = 1$, the growthrate is maximised when the labels $n$ of the two Kelvin modes are equal. The instability mode is in this case called principal and can be identified by the triplet $(m_1, m_2, n)$.

These results were later extended to vortices with continuous vorticity profiles. Le Dizès & Laporte (2002) give an expression for the growthrate in a pair of Gaussian vortices (co- and counter-rotating) without axial flow. They found that the most unstable modes are necessarily a combination of two Kelvin modes of azimuthal wavenumbers -1 and 1. This is coherent with the experimental results of Leweke & Williamson (1998) who studied the instability in a pair of counter-rotating vortices and observed an unstable periodic displacement of the vortex centres.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{(a) Visualisation of the elliptic instability on the top vortex vortex. (b) Schematic showing the characteristic scales.}
\end{figure}
Lacaze et al. (2007) studied the instability of a Batchelor vortex in a stationary strain field. As for the counter-rotating case, they found that the triadic resonance of the two Kelvin modes with the strain field was still leading to instability. Interestingly, they show that when the axial velocity amplitude increases, the \((-1,1)\) modes get damped. There exists a value of the axial flow parameter \(W_0\), above which the modes \((-1,1)\) are not the most unstable. Instead, modes with a more complex spatial structure, such as \((-2,0)\) can be observed.

### 3.3.2. Experimental observations

It can be seen in figure 6, that a short-wavelength core oscillation is present on top of the Crow instability. The aim of this part is to analyse in detail this perturbation. We focus here on the top vortex.

To observe the perturbation more closely, the laser beam was oriented in the upstream direction through the visualisation window (see figure 1). Fluorescein dye was injected into the top vortex and illuminated by the laser beam, and a camera was placed on the side of the test section, aiming at the vortex in the \(y\) direction. Figure 8(a) shows a periodic perturbation propagating on the top vortex. It is symmetric with respect to the axis of the vortex. The well-organised structure that can be seen in figure 8(a) was not observable on all frames. Only 80 frames (of 300 taken) could be analysed for \(z/c = 9.0\) to extract a wavelength \(\lambda_1\). Averaging the 30 values of the wavelengths measured, we obtain \(\lambda_1 = 1.60\) cm with a standard deviation of 0.15 cm. The axial wavenumber \(k_1 a_1 = 2.21 ± 0.14\).

In order to estimate the growthrate of these short-wavelength oscillations, measurements of the amplitude \(D\) of the perturbation defined in figure 8(b), were performed at \(z/c = 7.2, 7.8, 8.4, 9.0\) and 9.7, on the top vortex. At each streamwise position, 200 frames were recorded. 20 of them, showing the instability, were selected and processed. The growthrate \(\sigma_1\) is then estimated by a linear regression, taking into account the 100 amplitudes measured (see figure 9). We find

\[
\sigma_1^* = \sigma_1 \frac{2\pi b^2}{\Gamma_1} = 0.9 \pm 0.09, \tag{3.14}
\]

with \(b\) and \(\Gamma_1\) taking the value corresponding to \(z/c = 5.6\) (see table 1). The uncertainty on
TABLE 2. Parameters of the experimental and numerical flow.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Experimental</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z/c = 5.6$</td>
<td>numerical flow</td>
<td></td>
</tr>
<tr>
<td>$a_1/b$</td>
<td>0.147</td>
<td>0.147</td>
</tr>
<tr>
<td>$a_2/b$</td>
<td>0.142</td>
<td>0.147</td>
</tr>
<tr>
<td>$a_w/a_1$</td>
<td>0.88</td>
<td>0.88</td>
</tr>
<tr>
<td>$a_w/a_2$</td>
<td>0.85</td>
<td>0.88</td>
</tr>
<tr>
<td>$\Gamma_1/\nu$</td>
<td>17000</td>
<td>16800</td>
</tr>
<tr>
<td>$\Gamma_2/\nu$</td>
<td>-16500</td>
<td>-16800</td>
</tr>
<tr>
<td>$W_{01}$</td>
<td>-0.41</td>
<td>-0.41</td>
</tr>
<tr>
<td>$W_{02}$</td>
<td>0.43</td>
<td>0.41</td>
</tr>
<tr>
<td>$\sigma_{2\pi b^2}/\Gamma$</td>
<td>0.90</td>
<td>0.96</td>
</tr>
<tr>
<td>$k_m$</td>
<td>2.21</td>
<td>1.75</td>
</tr>
</tbody>
</table>

Figure 10. (a) Non-dimensional growthrate as function of axial wavenumber. The solid line represents the numerical result. The dot symbolises the experimental point. The mode structure corresponding to $k_1a_1 = 1.75$ and 3.2 are presented in (b) and (c).

the measurements of the amplitude $D$ is high: typically, the standard deviation reaches 30% of the mean value. This is due to the fact that the Crow instability is fully developed at this downstream location, generating a high variance of the separation distance $b$. Nevertheless, the resulting uncertainty† of the growthrate $\sigma_1$ is only 7%.

To link these experimental results with a known instability mechanism, a linear stability analysis was performed numerically using the full-spectral code presented in Roy et al. (2008b). The base flow investigated was chosen to be very similar to the flow measured at $z/c = 5.6$. At this location, the roll-up phase of the vorticity sheet is terminated and the amplitude of Crow’s instability and of vortex meandering are still reasonably small. The short-wavelength perturbation is not visible so close to the wings. However, the wavelength selection is expected to occur in this region. The base flow to be analysed was generated by solving the two-dimensional Navier–Stokes equations, starting from two axisymmetric vortices with Gaussian vorticity profiles and

† The evaluation of the uncertainty on the computation of $\sigma_1^*$ results directly from the least square analysis. It is described, e.g., by Taylor (1997).
circulation $\Gamma$. After a short period of oscillation during which the vortices relax to adapt to their mutually induced strain, a quasi–steady mean state is reached (see Le Dizès & Verga 2002). The flow obtained is a quasi–solution of Euler’s equations. It is then possible to stop the diffusion process and perform a stability analysis of the flow with respect to the three-dimensional perturbations. The vorticity radius $a$ of the flow is measured by fitting the velocity field with the two–dimensional velocity field of a pair of axisymmetric vortices. The axial velocity is introduced artificially by the relation

$$U_c(\theta, r) = \frac{\Gamma}{2\pi a} \left[ \frac{\xi_c(\theta, r)}{\xi_{c0}(0, 0)} \right]^{a^2/a_0^2},$$

where $U_c(\theta, r)$ and $\xi_c(\theta, r)$ are the two–dimensional fields of axial vorticity and axial velocity in the cylindrical frame centred on each vortex. The parameters of the numerical base flow are presented in table 2. The numerical growthrate is plotted in figure 10(a) as function of axial wavenumber. The most unstable zone, located around $k_1a_1 = 1.75$, corresponds to the elliptic instability. It was identified as the mode $(0, 2, 1)$ (see 3.3.1). The spatial structure of this mode is presented in figure 10(b). This unstable mode was theoretically predicted by Lacaze et al. (2007) in a pair of counter-rotating Gaussian vortices. The second–order azimuthal symmetry of this mode is coherent with the structure of the experimental perturbation observed, which is symmetric with respect to the vortex axis. The second most unstable zone, around $k_1a_1 = 3.2$, has an azimuthal symmetry of the third order, as shown in 10(c), and is therefore not a candidate to match the visualisations. The difference between the non–dimensional wavenumber $k_m$ corresponding to the largest growthrate and the experimental wavenumber $k_1a_1$ is 20%. This discrepancy is most likely due to the difficulty of identifying the base flow responsible for the wavelength selection. Indeed, the wavelength of the instability is determined by the conditions corresponding to the early stages of the flow. Practically, this means that the location $z/c = 5.6$, at which the base flow was measured, might already be too far downstream of the wing. A more upstream position would yield a smaller vorticity radius $a_1$ (the radius between $z/c = 5.6$ and $z/c = 9.4$ increased by 7%). Also, considering the mode $(0, −2, 1)$ for $W_0 > 0$, $k_m$ increases with $|W_0|$, as shown by Lacaze et al. (2007) and Roy et al. (2008b). Underestimating $U_c$ would therefore lower the wavenumber. The value of the experimental growthrate matches correctly the numerical one, given the experimental uncertainty.

Further details of the spatial structure of the instability were obtained from a different set of visualisations, with a laser sheet oriented vertically, normal to the free–stream. Images of the top vortex were taken through the visualisation window presented in figure 1, in a direction normal to the laser sheet; figure 11(a) shows an example. The Phantom high–speed camera was used to acquire 4000 frames at a rate of 300 Hz. The next step involved the so–called Proper Orthogonal Decomposition (POD) of this visualisation image series. Reviews of the POD method can be found in Berkooz, Holmes & Lumley (1993) and Liang, Lee, Lim, Lin, Lee & Wu (2002). A didactic approach is shown in Chatterjee (2000). In a previous study on the dynamics of a single wing tip vortex, it was shown that, concerning the qualitative spatial structure, POD of a series of dye images gave identical results as a POD of vorticity fields, which is physically more relevant (Roy & Leweke 2008, see also the Appendix). In the present study, the dye POD was used for its ease of implementation, in order to extract the mean features of the short–wave instability.

The method used for our decomposition can be explained as follows: the 8000 pixels of each frame were aligned to form one column of a matrix $F$. A singular value decomposition of $F$, which is of the form,

$$F = \hat{U} \Sigma \hat{V},$$

was then performed. In equation (3.16), the diagonal elements $S$ of $\Sigma$ are the singular values of
Figure 11. (a) Example of a visualisation used to perform the POD analysis on the light intensity. (b) Mode 5 and (c) mode 6 of the POD analysis. The dotted line is a circle of radius $a_1$.

Figure 12. Singular values $S$ computed from the light intensity of 4000 pictures of the top vortex.

$F_j$ arranged in decreasing order. All other elements of $\Sigma$ are zero. The columns of $\tilde{U}$ are the corresponding singular vectors. They are orthonormal and form a basis for the original frames. This basis is optimal in the least-square sense. For any integer $n_i \leq 4000$, it is not possible to find a basis in which the approximation of the frame series is better than the approximation with the first $n_i$ singular vectors. We call $n_s$ the rank of a singular value in the decreasing hierarchy. The first
Experiments on the elliptic instability in vortex pairs with axial core flow

FIGURE 13. The squares of the Fourier coefficients $C_5$ and $C_6$ are plotted for mode 5 and 6, as a function of $r$. The solid, thin and dashed lines correspond to $m=2$, $m=3$ and $m=0$. The other branches for which $m<10$ are too weak to appear on the figures.

singular values of $F$ are presented in figure 12†, showing, as expected, a decrease of the variance (or energy) of the modes with increasing rank. For our study, we focus on the modes number 5 and 6, $M_5$ and $M_6$, whose singular values are approximately equal. They are presented on figure 11(b) and 11(c). A second-order azimuthal symmetry can be visually identified. To analyse precisely their azimuthal dependence, for each mode, a Fourier decomposition was accomplished on the azimuthal direction for each radial position $r$. For each value of the wavenumber $|m|$, we obtain the Fourier coefficients as functions of $r$. The results of these computations are shown in figure 13. In figure 13(a), since no other branch reaches the order of magnitude of the branch $|m|=2$, $M_5$ can be considered as a pure spatial wave with an azimuthal wavenumber $|m|=2$.

Similarly, figure 13(b) shows two dominant branches at the highest order revealing the coupling of two spatial waves of azimuthal wavenumber $m=0$ and $|m|=2$. $M_6$ can therefore be considered as the superposition of two azimuthal waves. The presence of the $m=0$ component is revealed in figure 11(c) by the amplitude at the centre that does not fall down to zero. We can write

$$M_5 = \hat{A}(r) \sin(2\theta + \phi_0)$$

$$M_6 = M_6 - \hat{C}(r) = \hat{B}(r) \sin(2\theta + \phi_0 + 2\phi),$$

where $\hat{A}$, $\hat{B}$ and $\hat{C}$ are radial functions and $\phi$ is the phase difference between $M_5$ and $M_6$, the $|m|=2$ component of $M_6$.

We can show that a travelling wave $Y = \hat{Y} \sin(2\theta + \omega_f t)$ expressed in the $(M_5, M_6)$ basis must read:

$$Y = \frac{\hat{Y}}{\hat{A}} \frac{\sin(-\phi_0 - \pi + \omega_f t)}{\sin(\phi)} M_5 + \frac{\hat{Y}}{\hat{B}} \frac{\sin(-\phi_0 + \omega_f t)}{\sin(\phi)} \tilde{M}_6$$

In (3.18), the factors of $M_5$ and $M_6$ are the coordinates of $Y$. Experimentally, it is possible to extract the projection $X_5$ and $X_6$ of the frames on $M_5$ and $M_6$. This is illustrated in figure 14.

† Since the POD was performed on the light intensity values, the units of the singular values presented in figure 12, as well as the Fourier coefficient and the power spectral densities presented in figures 13 and 15, are not physically relevant.
for about 0.1 second. At the order, $X_5$ and $X_6$ can be seen as periodic, with a constant phase difference $\chi$, a similar amplitude $\tilde{X}$ and a zero mean-value. They can be approximately modeled by

$$X_5 = \tilde{X}(r) \sin(\omega_f t + \chi_0)$$

(3.19a)

and

$$X_6 = \tilde{X}(r) \sin(\omega_f t + \chi + \chi_0),$$

(3.19b)

According to equation (3.18), if $X_5$ and $X_6$ are to be associated with the coordinates of a travelling wave in the $(M_5, \tilde{M}_6)$ space, their relative phase difference must be such that $\chi = \phi \pm \pi$. Experimentally, we can measure the angular phase difference $\tilde{\phi}$ between $M_5$ and $M_6$ by a cross-correlation. We find $\tilde{\phi} = 45.00^\circ$ so $\phi = 2\tilde{\phi} = 0.5\pi$. By measuring the temporal shift $\tau$ of $X_6$ relatively to $X_5$, and their period $T$, we obtain $\chi = 2\pi\tau/T = 0.58\pi$. Given the approximation on the measure of $\tau$ due to the low temporal resolution of $X_5$, we can conclude that $X_5$ and $X_6$ are good candidates to be the coordinates in $(M_5, \tilde{M}_6)$ of a spatio-temporal travelling wave. This is illustrated and confirmed by the figure 14(b). It shows the spatio-temporal reconstruction of the field formed by putting side by side the successive projections of the frames on the modes 1, 5 and 6. Mode 1 can be considered as the mean field composed by averaging all the light intensity of the frames. Figure 14(b) shows a double-helix structure, typical of a second-order azimuthal
symmetry wave. The \( m = 0 \) component of \( M_6 \) just adds a periodic axisymmetric deformation to the helix, hardly visible on figure 14(b). Using the correspondence between the vorticity and the light intensity of the frames demonstrated in the Appendix, we can extend these conclusions to the vorticity field. A physical interpretation can be to consider perturbation field of the total field presented in 14(b), as the superposition of two Kelvin modes of pulsation \( \omega_f \) with different azimuthal wavenumbers \( m = 0 \) and \(|m| = 2\).

We are now wisht to make a link between these conclusions and the elliptic instability theory. A first step is to measure \( \omega_f \) more accurately. To do so, a Fourier analysis of the coordinates, in \((M_5, M_6)\), of all the frames acquired, is achieved. The corresponding power spectral densities \( P_5 \) and \( P_6 \) are plotted in figure 15. A low frequency peak can be clearly seen for \( P_5 \) and \( P_6 \). This is due to the Crow instability coupled with the meandering phenomenon. These two mechanisms impose a translation of the vortex that can be modeled by an \( m = 1 \) perturbation, but their high energetic levels also pollute the more complex modes. A frequency range of more interest is found between 25 Hz and 45 Hz. As can be seen in figure 15, it corresponds to the zone where the distance between \( P_5 \) and the linear approximation of \( P_5 \) in the middle range frequency, is maximal. The same applies to \( P_6 \). This gives the approximation \( \omega_f / 2 \pi = (35 \pm 10) \) Hz. Non-dimensionalising by \( 2 \pi b^2 / \Gamma_1 \), we find

\[
90 < \omega_f^* < 160.
\]

The non-dimensional pulsation of a spatial double-helix of wavelength \( \lambda_1 \), translating without any rotation at the velocity \( U_1 \) is equal to 135. This is the same order of magnitude as \( \omega_f^* \). It is consistent with the pulsation value obtained numerically which is one order smaller (see table 2). The spatio-temporal reconstruction presented in 14, can therefore be interpreted as purely spatial by making the transformation \( t \rightarrow -U_\omega t \). Since \( \lambda_1 \) is positive, we can then conclude on the positive sign of \( m \) by inspecting the angular velocity phase of the wave. The equivalent axial wavelength \( \lambda_f \) reads

\[
\lambda_f = 2 \pi U_\omega / \omega_f = 1.57,
\]
which is very close (2\%) to the value of $\lambda_1$ measured (1.60 cm).

We can now conclude on the identification of the elliptic instability in the present flow. We identified a periodic unstable perturbation. We decomposed this perturbation and showed that it was the result of the coupling of a spatial wave of azimuthal wavenumber $|m| = 2$ with an $m = 0$ (axisymmetric) wave. The difference between the two wavenumbers is equal to 2, which is typical of the elliptic instability (see 3.13). Despite a difference in the axial wavelength, the structure of the unstable mode, as well as the growthrate, are consistent with those of the most unstable mode obtained numerically by analysing the stability of the experimental base flow.

4. Co-rotating vortex pair

In this chapter, we investigate the stability of a co-rotating vortex pair with axial core flow. Such pairs were generated in the water channel by changing the sign of the angle of attack of one of the wings, with respect to the configuration used in the previous section. The values of free-stream velocity and angles of attack, as well as the relative position of the two airfoils, were varied until a short-wavelength perturbation could be observed on the vortices. One such configuration was then selected for a more detailed analysis. It consisted of the two wings being offset in the horizontal direction ($x$), with tips of their trailing edges located at the same vertical position ($y$). The angle of attack was 8° for both wings, and the tips were separated by 2.5 cm in the $x$-direction. The free-stream velocity was near 65 cm/s, resulting in a chord-based Reynolds number of about 70000.

4.1. Characteristics of the three-dimensional base flow

Figure 16 shows a side view of a dye visualisation of the co-rotating vortex pair. Mutual induction of the two vortices induces a rotation of the pair as it moves downstream (towards the right) in the channel, which produces the double helix configuration visible in figure 16.

Following the procedure described for the counter-rotating pair, 300 velocity fields were measured at $z/c = 9.0$, using Stereo-PIV. The same recentering method was used to remove the effect of the vortex motion on the time-averaged velocity fields. Figure 17 shows a reconstruction of the total mean field. For negative values of $x$ (to the left of the middle plane), the fields were recentered relatively to the top vortex and then averaged. The same was made relatively to the bottom vortex to compute the rest of the field. Considering one vortex, the stretching direction of the average vorticity distribution follows approximately the principal direction of the strain field induced by the other vortex. Considering the axial velocity distribution, the stretching direction is governed by the main part of the wake of the wing, which spirals around the vortices. For each vortex, the total circulation was estimated by integrating the velocity field on a closed rectangular contour surrounding the vortex, as for the counter-rotating case. The vortices were found to be, to a good approximation, equal in strength.

To model the azimuthally averaged azimuthal velocity profile $U_\theta$, contrary to the counter-rotating case, Gaussian functions did not lead to acceptable results. Instead, following the approach of Fabre & Jacquin (2004), the measured profile $U_\theta$ was fitted to their VM2 vortex model, according to:
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\[ U \text{ (cm/s)} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17}
\caption{Reconstituted time-averaged vorticity (a) and axial velocity (b) fields at \( z/c = 9.0 \). The top vortex is now on the left.}
\end{figure}

\[ U_\theta = \frac{\Gamma \rho_1^{-1}}{2\pi \rho_1} \frac{r}{[1 + (1/\rho_1)^4]^{1+\eta/4} [1 + (1/\rho_2)^4]^{1-\eta/4}} \]

where \( \rho_1 \) and \( \rho_2 \) correspond to two different characteristic radii of the vortex. \( \rho_1 \) delimits the inner viscous core containing most of the vorticity, and \( \rho_2 \) defines the region containing all of the circulation. For many types of wing tip vortices, \( \rho_2 \) is significantly larger than \( \rho_1 \). The match is excellent, as shown in figure 18(a). The parameters obtained for each vortex are listed in table 4.1. The two vortices are very similar to each other. Similarly to the counter-rotating case, \( U_{\infty} \) is defined as the value of the axial velocity profile at \( r/\rho_1 = 3 \). The non-dimensional parameter \( W_0 \) is in this case equal to \( W_0 = (U_0 - U_{\infty})2\pi\rho_1/\Gamma \). \( W \) was very well fitted with a Gaussian
TABLE 3. Parameters of the base flow extracted from Stereo-PIV measurements. The azimuthally averaged azimuthal velocity profile was fitted with a VM2 model defined by 4.1. Indices 1 and 2 refer to the upper and lower vortex, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Top Vortex</th>
<th>Bottom Vortex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>-169 cm$^2$/s</td>
<td>-170 cm$^2$/s</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>0.45 cm</td>
<td>0.44 cm</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>2.14 cm</td>
<td>2.30 cm</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.62 cm</td>
<td>0.59 cm</td>
</tr>
<tr>
<td>$a_w$</td>
<td>0.47 cm</td>
<td>0.46 cm</td>
</tr>
<tr>
<td>$b$</td>
<td>2.69 cm</td>
<td></td>
</tr>
<tr>
<td>$\Gamma/\nu$</td>
<td>18000</td>
<td>18100</td>
</tr>
<tr>
<td>$\rho_1/b$</td>
<td>0.17</td>
<td>0.16</td>
</tr>
<tr>
<td>$a_w/\rho_1$</td>
<td>1.04</td>
<td>1.05</td>
</tr>
<tr>
<td>$W_0$</td>
<td>0.33 cm</td>
<td>0.29 cm</td>
</tr>
</tbody>
</table>

FUNCTION (see figure 18). $W_{01}$ is observed to be 12% greater than $W_{02}$. This difference is mainly due to a higher axial velocity defect in the centre of the top vortex compared to the bottom vortex, since the top wing is longer than the bottom wing.

In order to characterise the evolution of the vortex positions with downstream distance $z$, a laser sheet oriented in a direction normal to the free–stream was placed at $z/c = 1.7, 4.8, 7, 8.8$ and 9.8. Dye was injected at both wing tips. For each streamwise location, 6000 frames showing vortices were acquired. For every frame, each vortex was localised by computing the centre of mass of the light intensity. Averaging the coordinates obtained for each $z/c$ gives a reliable estimation of the vortex positions in the $(x,y)$ plane. This is illustrated in figure 19. It is clear that the vortex pair moves upward as $z/c$ increases. A possible explanation for this behavior could lie in the large–scale background flow (rotation), induced by the wings inside the test section, which may entrain the vortex system away from its initial $x-y$ position.

The separation distance $b$ evolved approximately linearly with $z/c$, varying from $b = 4.2$ cm at $z/c = 4.8$, to $b = 2.4$ cm at $z/c = 9.8$. A similar linear evolution was observed for the angle $\theta$. 

![Figure 18](image-url)
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4.2. Elliptic instability

Following the procedure used for the counter-rotating pair, fluorescein dye was injected into the top vortex, illuminated in volume in a direction parallel to the free-stream. At 

\[
\frac{z}{c} = 9.5, \quad 200
\]

frames of the top vortex were captured. 15 of them, exhibited a short-wave perturbation with axial wavelength \(\lambda\), an example is shown in figure 21. Averaging the values measured, we find

\[
\lambda_1 = (1.09 \pm 0.12) \text{ cm.} \tag{4.2}
\]

which corresponds to the non-dimensional axial wavenumber

\[
k_1 \rho_1 = 2\pi \rho_1 / \lambda_1 = 2.59 \pm 0.36. \tag{4.3}
\]
Since this value is about twice the vorticity radius $\rho_1$, it scales with the elliptic instability expectations. The perturbation is symmetric with respect to the axis of the vortex. This implies an even order of azimuthal symmetry. Since it was not visible upstream of $z/c \approx 9$, this perturbation is unstable and was triggered by the flow.

The theory of the elliptic instability presented in 3.3.1 for counter-rotating vortices (stationary strain field), also applies to co-rotating vortices. The rotation of the mutually induced strain field at the rate $\Omega$ changes the resonance condition (3.13) to

$$
\begin{pmatrix}
  k_1 \\
  m_1 \\
  w_1
\end{pmatrix} -
\begin{pmatrix}
  k_2 \\
  m_2 \\
  w_2
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  2 \\
  \Omega
\end{pmatrix}.
$$

(4.4)

Following these conditions, if the growing perturbation we observed was to be related to the elliptic instability, it would necessarily be the result of a resonance between Kelvin modes with $m_1 = 0$ and $|m_2| = 2$ or $|m_1| = 2$ and $|m_2| = 4$. Higher-order mode combinations would hardly be visible experimentally, local minima being partially hidden by maxima shifted by an angle of $\pi/4$, in the azimuthal direction. Figure 22 qualitatively compares the wavelength obtained in the experiment to the numerical results of Roy et al. (2008b) who studied the elliptic instability in a pair of Gaussian co-rotating vortices with axial flow. The experimental point is close to the mode $(0, -2, 2)$. For the value of $W_0$ measured experimentally, this mode is the most unstable one. The spatial structure of this mode is consistent with our observations. Given the fact that the base flow analysed is different in the two studies, the comparison remains qualitative, but it suggests that the $(0, -2, 2)$ mode is likely to be the cause of the instability observed experimentally.

5. Conclusions

In this paper, we have presented experimental results concerning the short-wavelength elliptic instability of a pair of co- and counter-rotating vortices with an axial velocity core. This instability was previously observed clearly only in vortex pairs without axial flow (Leweke & Williamson 1998; Meunier & Leweke 2005). It was demonstrated here to persist with the addition of the axial velocity. Dye visualisations and stereoscopic particle image velocimetry measurements allowed to extract qualitative and quantitative information on the spatial structure of the instability mode observed.

The axial wavelength of the short–wave instability was measured experimentally from dye
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visualisations of the vortex pairs. For the counter–rotating case, it was found to be in good agreement with the results of a linear stability analysis performed numerically on the experimental base flow. The discrepancies can be explained by the difficulty to determine the base flow responsible for the wavelength selection. The instability is observable at a given downstream position, where its amplitude is high enough to be noticed, but it was also on the vortex upstream of this position. The exact location where it was initiated, i.e. where it is selected, difficult to determine.

For the counter-rotating case, the experimental growth rate of the instability was determined from measurements of the amplitude at different downstream locations. The result matches very well with the numerical growth rate of the most unstable mode the elliptic instability.

To investigate in detail the spatial structure of the observed unstable mode, a Proper Orthogonal Decomposition based on dye visualisations of the top vortex was carried out. It revealed the superposition of two three-dimensional waves of similar magnitude, that can be related to Kelvin modes, both periodic in the axial direction. The first wave was axisymmetric, corresponding to an azimuthal wavenumber $m = 0$; the second one had a second order azimuthal symmetry ($m = 2$). The difference of 2 in the azimuthal wavenumber is typical of the elliptic instability in a vortex pair. Moreover, the azimuthal symmetry of the two waves corresponds exactly to the azimuthal symmetry of the two Kelvin modes resonating to form the most unstable mode obtained from numerical stability analysis.

In summary, we have, in this study, presented clear evidence of a short–wavelength three–dimensional instability, developing in counter- and co-rotating vortex pairs with axial flow jet flow in their cores. The spatial structure of the unstable perturbation is characterised by an azimuthal wavenumber $m = 2$, resulting in a double–helix structure of the perturbed vortex core. This is different from the short–wave instability modes previously observed on vortices without axial flow, which had an azimuthal variation with $m = \pm 1$. Qualitative and quantitative comparison of the experimental results with theoretical/numerical stability analysis have clearly identified this phenomenon as an elliptic instability of the vortex cores, caused by the mutually induced strain of the vortices.

The authors wish to thank S. Le Dizès for many helpful discussions during the course of this

Figure 22. Contours of instability growth rate in the ($W_0, ka$) plane for $a/b = 0.14$ and $Γ/ν = 88000$. This figure concerning a pair of co-rotating Gaussian vortices with axial core flow, was taken from Roy et al. (2008b).
study. This work was supported by the European Commission under Contract no. AST4-CT-2005-012238 (FAR-Wake).

Appendix. Relation between vorticity and dye POD modes

In this part, we demonstrate that the modes extracted by Proper Orthogonal Decomposition or POD (Berkooz et al. 1993; Chatterjee 2000; Liang et al. 2002) on the vorticity field of a single vortex, show the same basic characteristics of the modes extracted by a POD on the light intensity of the frames where the vortex can be spotted by a dye patch.

For this work, presented in Roy & Leweke (2008), the same setup as described in §2 was used. Only the bottom wing, which can be seen in figure 2(b), was conserved in the water channel, to generate a single tip vortex in the free–stream. The angle of attack was fixed to 6° and the free–stream velocity to 46.6 cm/s. 400 Stereo-PIV measurements were performed to extract the three components of the velocity in a plan normal to the free–stream at $z/c = 11.2$. Following the recentring method presented in §3.1, the axisymmetric azimuthal velocity profile $U_\theta$ was fitted by the two–scale vortex model, VM2 defined by (4.1). The match is excellent. The axial velocity defect profile was fitted to a Gaussian function. The flow was characterised by the non-dimensional parameters listed in table 4.

The best way to compare accurately the results of the POD analysis on the vorticity and the light intensity is to run the POD on the same instantaneous flow for the two procedures. To make this possible, in addition to the Dantec particles used previously for the Stereo-PIV measurements, some fluorescein was injected at the wing tip and advected with flow. At $z/c = 11.2$, an argon ion continuous laser illuminated the flow in a vertical plane normal to the free–stream. A cylindrical lens was used to focus the laser power on a small zone containing the vortex. A high–speed video camera was positionned at the end of the test section, watching the laser plane.
through the visualisation window presented in figure 1. For a duration of 8 seconds, 16000 images were recorded at an acquisition rate of 2000 Hz. The acquisition rate was fixed by the PIV requirement. On each frame, the particles seeding the flow and a dye patch similar to the one presented on figure 11 were visible. The light intensity level reflected by the particles was higher than the dye patch allowing the measure of the two component velocity field using planar PIV. For each frame, the vorticity was computed. The POD procedure based on singular value decomposition described in §3.3.2 was followed to extract the set of optimal modes for the vorticity and the light intensity. The singular values are presented in figure 23. The corresponding modes can be found in figures 24 and 25. The dimensions of the zone plotted in all the modes presented in figures 24 and 25, and the scale of the axes are the same. Also, the colour scale is symmetric with respect to 0, from black to white. Figures 23(a) and 23(b) look qualitatively alike. The singular values, which can be seen as the energy of the corresponding singular vector, rapidly decrease with the increasing order. The first mode corresponds to the mean field. In both cases, they show an axisymmetric zone (to the first order) around which the relative magnitude always has the same sign. The energy of this mode is much higher than the others. Modes 2 and 3 are, in both cases, associated with a global displacement of the vortex in the plane. They can be seen as a pure azimuthal wave of wavenumber \( m = 1 \). The only difference that can be noticed is a small tilt of the dye modes compared to the vorticity modes. This can be explained by the fact that the preferred direction of motion of the vortex is not very sharply defined as shown by Roy & Leweke (2008) for this flow. Nevertheless, the relative orientation of modes 2 and 3 remains constant at 90°. In figure 23(a) and 23(b), modes 2 and 3 form a doublet of similar energy, well separated from the rest of the modes. Both vorticity and dye POD identify the vortex displacement as the most energetic perturbation. Modes 4, 5 and 6 also look very similar. They all seem to result from

**Figure 24.** Modes computed by singular value decomposition of the vorticity field time series obtained by high-speed PIV for a single vortex (Re = 8700 and \( W_0 = 0.27 \)). The six most energetic modes are shown.
the combination of $m = 2$ and $m = 0$ waves. Their absolute and relative directions are similar. The vorticity modes 4, 5 and 6 are slightly more isolated in the singular value hierarchy than the dye modes. They form a triplet that is easily visible in figure 23(a). Nevertheless, the rank of the modes is the same.

As a conclusion, the match between vorticity and light intensity modes obtained by POD is excellent. The same spatial structures were revealed in both cases. Furthermore, the hierarchy in the energy levels is conserved. This leads to the powerful conclusion that a POD analysis can be performed on dye visualisations to obtain information on the vorticity modes.

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