Temporal stability analysis of 4-vortex systems by Floquet methods

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Highly resolved solutions of the two-dimensional incompressible Navier-Stokes and continuity equations, describing the evolution of a co-rotating pair of vortices, have been obtained accurately and efficiently by spectral collocation methods and an eigenvalue decomposition algorithm,$^1$ as well as the Nektar$^2$ spectral–$hp$ code. Excellent agreement has been obtained with referenced results.$^3$

Such solutions have formed the basic states for subsequent three-dimensional Floquet eigenvalue problem linear instability analyses. This approach monitors the modal response of time-periodic vortical systems to small-amplitude perturbations, periodic along the homogeneous axial spatial direction, without the need to invoke an assumption of azimuthal spatial homogeneity, but taking the temporal periodicity of the basic state into account.

The numerical work has been performed by adapting the Floquet module of Nektar in order to study instability of time-periodic vortical flows. Analyses of two pairs of vortices, as well as the entire flowfield comprising a four-vortex system, have been performed. The extremely long computational times required in order to reduce residuals in the Floquet multipliers by three-to-four orders of magnitude, before considering the results as physically relevant, have limited the number of parametric studies performed. Preliminary results obtained suggest that the periodic orbit defined by the basic states analyzed is unstable against two-dimensional perturbations.

I. Introduction

Finite element methods were introduced in fluid mechanics in the late 1970s and are used routinely in large-scale flow simulation codes since. They have proven very successful in simulation of flows in applications in which geometric complexity makes alternative methods less efficient. Besides large-scale computation, information on 3-d flow dynamics may be obtained by (global) instability analysis. In particular, within a instability context the flow analyzed develops inside or over arbitrarily-shaped two dimensional domains, homogeneous in the third spatial direction. While the field of instability analysis of steady two-dimensional, essentially non-parallel flows, coined BiGlobal analysis, is rapidly maturing,$^1$ Floquet methods applied to time-periodic such flows are substantially less widespread, primarily due to the algorithmic complexities and ensuing computing cost associated with them. Starting with the seminal work of Barkley and Henderson,$^4$ practically all Floquet stability analyses performed to-date have employed high-order numerical discretizations. The reason clearly lies in the fact that (as with any stability analysis) a high-order method permits recovery of accurate instability results with a reasonable computing cost. This is especially true as the Reynolds number increases and increasingly finer structures appear in the fluid. Unlike classic stability analysis, where mesh refinement can be used at will, in a Floquet analysis context in which the discretized operator is kept in memory, the resulting computing requirements are prohibitively high. The alternative approach is to resolve the structures formed in the fluid with a high-order polynomial approximation.

The present contribution focuses on Floquet analyses of the quasi-periodic basic flow resulting from two pairs of co-rotating vortices. The underlying assumption is that such a flow defines a periodic orbit and
the objective of the analysis is to quantify its stability against two- and three-dimensional perturbations. A related question involves the use of the "temporal" DNS model, in which axial periodicity is imposed, in order to recover the basic flows to be analyzed. Recent evidence suggests that the instability behavior of a spatially- and temporally-obtained basic states may be quite different.\textsuperscript{5} From a purely technical point of view, it is interesting to assess the computational cost of Floquet methods, as applied to the problem at hand, for which the interesting Reynolds numbers are at least an order of magnitude larger than the bluff-body flow problems on which this theory has delivered excellent comparison with experiment (e.g.\textsuperscript{4}), is presently unclear.

Section II discusses the theoretical background of both the basic flow and the Floquet analysis methodology, as implemented in the well-tested Nektar\textsuperscript{2} code. Subsequently, results are presented in section III, on the quantification of the basic states analyzed and the instability thereof. Owing to the extremely expensive nature of the work, only a limited number of runs has been performed. This has nevertheless been sufficient in order for first conclusions to be reached. The latter are discussed in section IV.

II. Theory

A. The Base Flow

The baseflow of a co-rotating pair of vortices may be modeled by the two-dimensional Navier-Stokes equations. Two different codes have been used for the baseflow calculations. First, a spectral/hp code based on the primitive variables $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}$ formulation has been used.\textsuperscript{2} The spectral/hp-element method combines attributes from both the $h$-type and the $p$-type extensions of a finite-element approach. Recognizing the advantages of both types of convergence,\textsuperscript{6} proposed and implemented a new method that he coined the $hp$ version of the finite element. In an $hp$–FEM discretization one applies a polynomial expansion of a given order within each elemental region. A favorable expansion is typically an orthogonal or near-orthogonal set of functions within the standard regions. A second spectral collocation code, well-validated within the present, as well as earlier EU-wide efforts on the trailing-vortex problem,\textsuperscript{7} has been used for cross-validation and eccentricity calculations. Key characteristics of the second approach are the vorticity/stream function formulation, which automatically satisfies continuity to machine level, as well as the efficient solution of the Poisson problems during each sub-step of the time-integration by eigenvector decomposition techniques.\textsuperscript{1} A Cartesian ($x_1, x_2, x_3$) coordinates system will be considered for all the baseflow calculations, taking $x_1$ to be the axial spatial direction.

The equations in primitive variables, as solved by the spectral/hp code are

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} \quad \text{in} \quad \Omega, \quad (1)$$

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega, \quad (2)$$

$$\tilde{u}_i = 0 \quad \text{on} \quad \Gamma_D, \quad (3)$$

$\Omega$ is the computational domain and $\Gamma_D$ is the domain boundary where homogeneous Dirichlet boundary conditions were applied. If boundaries are far enough the basic flow calculations are not affected by its presence. The time evolution of two identical vortices will be considered. Initially both vortices are perfectly axisymmetric according to the Gaussian profile:

$$\omega_0(r) = e^{-r^2} \quad (4)$$

This Gaussian vortices centered on $(0, \pm x_n)$ can be described in primitive variables and Cartesian coordinates as:

$$u = -qy \left( \frac{1 - e^{-\left(x-x_n\right)^2+y^2}}{(x-x_n)^2+y^2} - \frac{1 - e^{-\left(x+x_n\right)^2+y^2}}{(x+x_n)^2+y^2} \right),$$

$$v = q(x-x_n) \left( \frac{1 - e^{-\left(x-x_n\right)^2+y^2}}{(x-x_n)^2+y^2} + \frac{1 - e^{-\left(x+x_n\right)^2+y^2}}{(x+x_n)^2+y^2} \right) \quad (5)$$

where $q = 0.5$ to match with the vorticity equation.
The non-dimensionalized initial circulation $\Gamma_0$ and initial radius $a_0^2$ of these vortices can be calculated as:

$$\Gamma_0 = \int \int_\Omega \omega d\tau = \pi$$

(7)

$$a_0^2 = \frac{\int \int_\Omega r^2 \omega d\tau}{\Gamma_0} = 1$$

(8)

The initial distance between both vortices is called $l_0$. According to this, we can define the initial aspect ratio of the vortex pair as $a_0/l_0$. The dynamics is mainly characterized by the Reynolds number, which is defined as $Re = \frac{\Gamma}{\nu}$.

In an inviscid point vortex model of two co-rotating vortices where the circulation $\Gamma$ of each vortex is concentrated in its centroid, the vortices remain separated by a constant distance $l_0$, and rotate around each other at a constant angular frequency $f_0 = \frac{1}{\pi l_0^2}$. Each vortex moves due to the velocity field induced by the other one according to the Biot-Savart law. As LeDizes and Verga\(^3\) point out, this description is also valid for finite-size vortices. They estimate the period $T_0$ of the co-rotating configuration as $T_0 = \pi b_0^2$, and the turnover time as $2T_0$. The period of the moving system, as well as the distance between vortices, remains constant until a critical time is reached. After that critical time, the angular frequency starts growing while the distance between vortices tends to decrease towards the final breakdown and merging process.

Considering finite vortices instead of 'point vortices', we can appreciate two physical phenomena that contribute to the dynamics of a two-vortex system: the first is the deformation of each vortex due to the velocity field induced by the other, creating a train field that tends to deform each vortex and to convert it into an ellipse,\(^8\) and the second is the viscous dissipative process.

The global elliptic deformation of the vortices can be studied by measuring the global eccentricity of the ellipses. This calculation in an appropriate rotating reference frame ($X, Y$) is analogous to that done for the counter-rotating pair of vortices.$^9, 10$ We can consider that up to the critical time vorticity has not been advected or diffused across the symmetry line that separates both vortices, this line is represented by the $y$ axis. Vortex radius $a$ and vortex eccentricity $\varepsilon$ are defined as:

$$a = \sqrt{\frac{a_M + a_m}{2}}$$

(9)

$$\varepsilon = \frac{a_M - a_m}{a_M + a_m}$$

(10)

where $a_M$ and $a_m$ are the larger and smaller vortex radii. Diagonalizing the inertia tensor results in the product of $\Gamma a_M^2 + \Gamma X^2$ and $\Gamma a_m^2$ as the bigger and smaller eigenvalues. This vortex centroid calculation is performed in a rotated coordinate system such that the vortex centroids are located at $(0, \pm X_c)$. The coordinate $X_c$ is calculated as:

$$X_c = \int \int_{HP} X \omega dX dY / \Gamma$$

(11)

The distance between vortices can be calculated as $b = 2X_c$. The global eccentricity gives an accurate measure of the deformation of the entire vortex region.

Similarly to the evolution of two counter-rotating vortices, the dynamics of the system can be clearly separated in two parts: first, a non-viscous adaptation of each vortex to the external field induced by the other, this process breaks the self-symmetry of each vortex giving a elliptical shape. As is shown in\(^3\) this adaptation is not uniform and the vortex core gets the elliptical shape much faster than other regions which have a more complex evolution. The time scale associated with this adaptation process is the non viscous time scale and the characteristic time $T_\nu = \frac{2\pi a_0^2}{\nu}$. After this there is a second viscous diffusion process, also appreciated in a linear growth of the aspect ratio $a/b$ which evolves on a viscous time scale with characteristic time $T_\nu = \frac{2\pi a_0^2}{\nu}$.

For the validation of the baseflow calculation a second spectral collocation code was also used where a Cartesian coordinate system was also considered, taking $(x_1, x_2, x_3) \equiv (x, y, z)$ and $(u_1, u_2, u_3) \equiv (u, v, w)$. The basic flow is calculated by time-marching the vorticity transport equation:

$$\zeta_t + \nu \zeta_{\phi} + \nu \zeta_{\phi} - \frac{1}{Re} \nabla^2 \zeta = 0,$$

(12)
where $\zeta = -\partial_y \nu + \nu_z$ is the basic flow vorticity, $\nabla^2 = \partial_y^2 + \frac{\partial^2}{\partial z^2}$, and the streamfunction, $\psi$, is related with the vorticity through

$$\nabla^2 \psi + \zeta = 0. \quad (13)$$

Dimensional time $t^*$ is non-dimensionalized as

$$t = \frac{t^* \nu}{2\pi a_0^2} \quad (14)$$

The initial conditions for the flows analyzed are composed of an isolated or system of Gaussian vortices. The initial vorticity $\zeta_0 = \zeta(t = 0)$ of a single such vortex is defined by

$$\zeta_0 = \frac{\Gamma_0}{\pi a_0^2} e^{-\frac{y^2}{a_0^2}}. \quad (15)$$

An eigenvalue decomposition algorithm$^1$ is used in order to solve efficiently the Poisson equations arising from temporal discretization of (12) by the high-order algorithm proposed by Spalart, Moser and Rogers.$^{11}$

By contrast to the case discussed by Theoﬁlis,$^1$ periodic boundary conditions are used here, as justiﬁed by the disparity of the characteristic length scales of the vortex system and the much larger integration domain. This results in a particularly simple algorithm for the solution of the Poisson problems, as outlined next.

A typical Poisson problem to be solved during each fractional time-step has the form

$$\nabla^2 f = g, \quad (16)$$

defined in a rectangular domain $y \in [-y_\infty, y_\infty] \times z \in [-z_\infty, z_\infty]$. The domain is discretized by $N_y$ and $N_z$ Fourier spectral collocation points,$^{12}$ whereby the corresponding collocation derivative matrices, $D_y^2$ and $D_z^2$, respectively denoting second order derivative with respect to the spatial variables $y$ and $z$ have been used. Diagonalizing $D_y^2 = A_k A^{-1}$ and $(D_z^2)^T = B \lambda B^{-1}$ one obtains

$$A_k A^{-1} f + f B \lambda B^{-1} = g \quad (17)$$

where $f = A^{-1} f B$ and $g = A^{-1} g B$. The system of algebraic equations (18) may be solved efficiently, i.e. requiring $O( (N_y^2 + N_z^2 ) \text{ memory and } O( (N_y + N_z)^2 )$ computing time for the forward and backward sweeps between $(f, g)$ and $(\hat{f}, \hat{g})$ at each fractional time-step, as opposed to $O( (N_y N_z)^2 )$ memory and $O( (N_y N_z)^3 )$ computing time that a direct solution of (16) would require.

Excellent agreement has been found between both codes, where the maximum error in the velocity components for some random snapshots comparison were lower than $10^{-6}$.

B. The Floquet Analysis

First of all some validations with different geometries were done just to make sure that some results obtained in classical problems were obtained with NekTAR code. A circular cylinder was run with a periodic baseflow, and afterwards the Floquet analysis was performed using 32 frames of the period, see figure 7. Similarly a Low Pressure Turbine (LPT) was also analyzed under periodic baseflow conditions, see 8.

The baseflow analysis has come with an artificial rotation of a frozen snapshot obtained from the evolution of the vortex pair in the viscous regime. This way at viscous time $t = 0.04$ was selected, and rotated for 32 equidistant angles of rotation which cover the whole period. This frozen period is a typical extension of what has been done with the counter-rotating pair in.$^{13}$ For other Floquet analyses like,$^{14}$ a really periodic baseflow can be obtained due to the fact that the energy is conserved in the whole domain, there is an inflow that acts as an energy supplier. In our case the energy is not conserved in the time evolution due to viscous effects and that means that some considerations have to be done to create a periodic baseflow.

Central to linear flow stability research is the concept of decomposition of any flow quantity into a steady or time-periodic laminar basic flow upon which small-amplitude, in principle multi-dimensional, disturbances are permitted to develop. In the context where the basic state is time-independent and homogeneous in its
third dimension this concept can be expressed by decomposing the solution, given in the following by equation 19 for the case of incompressible flow.

\[ u_i = \tilde{u}_i(y, z) + \epsilon \tilde{u}_i, \]  
\[ p = \tilde{p}(y, z) + \epsilon \tilde{p}, \]  

where \( \epsilon \ll 1 \) and \( \tilde{u}_i, \tilde{p} \) are small-amplitude the flow perturbations, for which

\[ \tilde{u}_i(x_i, t) = \bar{u}_i(x_i; t + T) \]  

holds. Here a temporal formulation has been adopted, considering \( \alpha \) is a real wavenumber parameter associated with the axial periodicity length through \( L_x = \frac{2\pi}{\alpha} \) and \( \omega \) is the complex eigenvalue sought.

Substitution into Navier-Stokes equations results in

\[ i \alpha \hat{q} + \hat{v}_y + \hat{w}_z = 0, \]  
\[ (\mathcal{L} \hat{u} - \bar{u}_y) \hat{v} - \bar{u}_z \hat{w} - i \alpha \hat{p} = \frac{\partial \hat{u}}{\partial t}, \]  
\[ (\mathcal{L} - \bar{v}_y) \hat{v} - \bar{v}_z \hat{w} - \hat{p}_y = \frac{\partial \hat{v}}{\partial t}, \]  
\[ (\mathcal{L} - \bar{w}_z) \hat{w} - \bar{w}_y \hat{v} - \hat{p}_z = \frac{\partial \hat{w}}{\partial t}, \]

where \( \mathcal{L} = 1/Re \left( -\alpha^2 + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - i \alpha \bar{u}_y - \bar{v}_y - \bar{w}_z. \)

Equation (22) can be expressed in the time-differential form as

\[ \frac{\partial \hat{q}}{\partial t} = A \hat{q}. \]  

In order to solve this equation in an efficient manner the Arnoldi algorithm, which is based on a Krylov subspace iteration method, has been used in combination with the exponential power approach that solves equation (26) explicitly

\[ q(t + \Delta t) = Bq(t) = q(t)e^{\int^t_{t-\Delta t} A \text{d} \tau}. \]  

Employing the Arnoldi algorithm on the evolution operator \( B(t) \), which evolves \( q(t + \Delta t) \) from \( q(t) \) yields the dominant eigenvalues of \( B = e^{\Delta t A} \), where we have initially assumed \( A \) to be independent of time. In order to employ the matrix exponential, the action of the operator can be represented through a time-stepping algorithm. In that “matrix-free” framework, neither the operator \( A \) nor \( B \) has to be constructed and only their actions are being considered, ultimately enabling us to investigate complex problems where the large size of \( A \) and \( B \) would otherwise be constraint by memory limitations.

The eigenvalues with the largest real part of the matrix \( \Delta t A \) are then determined. Since only the stability-significant leading eigenvalues are calculated, the run-time associated with the total process of building the Krylov subspace and obtaining the eigenvalues of the Hessenberg matrix constructed by the iteration is a small fraction of that required by classic methods, such as the QZ algorithm.

In a Floquet context, we are interested in the stability about periodic orbits rather than in the development of perturbed steady state base flows. The modified algorithm necessary to accomplish this task monitors the T-periodic basic state which is then considered to be of the form

\[ \tilde{q} = \bar{q}(x, y, t) = \bar{q}(x, y, t + T). \]  

As the Jacobian matrix is not steady but time-periodic, the stability can not be determined by the eigenvalues of \( A \). Rather, stability is determined by the eigenvalues \( \mu \) of the monodromy matrix operator \( B \) now defined as

\[ B = e^{\int^T_0 A \text{d} \tau}. \]
Equation (29) can heuristically be understood as how linearized perturbations evolve around one period \( T \) (where the periodic orbit is discretized in, say, \( N_T \) snapshots). For that reason one has to be aware of the situation where growth and decay can both occur within one cycle and that by integrating over \( A(t) \) only the average growth will be determined. The eigenvalues \( \mu \) are called Floquet multipliers. \( \mu > 1 \) describes a growing orbit, \( \mu < 1 \) leads to a limit cycle, respectively, see\textsuperscript{15} for further details.

III. Results

A. Basic flow

For reasons that will become clear in what follows, the four-vortex problem has been approached in a step-by-step manner, first analyzing one pair of co-rotating vortices (and its counterpart) before addressing the full system of four vortices. According to LeDizes and Verga\textsuperscript{3} Reynolds number \( Re = 8000 \) is high enough to have well separated viscous and convective time scales. The selection of the aspect ratio \( a_0/b_0 \) must comply with the criteria of this ration being small enough to avoid the merging process threshold and, at the same time, large enough so that the period of rotation

\[
T_0 \sim b_0^2
\]

is sufficiently short in order to make the instability analysis computationally feasible.

The simulations were performed in a square box of size \( L \). With the finest code, spectral/hp Nektar, an unstructured grid of 4578 elements(3200 quadrilaterals+1378 triangles) with an \( 8^b \) polynomial order approximation for the modal spatial discretization. As it was said before, the size of the box \( L = 144 \) has been chosen big enough to allow homogeneous Dirichlet boundary conditions. In figure 1 mesh and geometry are shown, it can be seen how an \( h \)-refinement has been done in the area where the vortex movement is taking place. The Floquet analysis has been performed using \( N_T = 32 \) equally spaced snapshots in order to represent each period of the basic flow.

Vortices are initially placed on the x-axis at \( x = \pm 2.5, (a_0/b_0 = 0.2) \), at \( Re = 8000 \); see figure 2, where the evolution of the vortex pair at \( t = 0, T_0/4 \) and \( T_0/2 \) is shown. The relative proximity of the vortices in this case leads to the development of the cat’s-eyes structure in the axial flow vorticity, as clearly seen in this figure. By contrast, figure 3 shows results of the same configuration, the vortices now initially placed at \( x = \pm 5, (a_0 = b_0 = 0.1) \). Besides the fact that the rotation period in the case \( a_0 = b_0 = 0.1 \) is approximately four times larger than that of the case \( a_0 = b_0 = 0.2 \) (a fact which will have consequences on the convergence of the Floquet multipliers, as shown later) one notices that there is significantly less interaction between the two vortices in the former case. In both cases, one can also see that the viscous diffusion process is well underway by the time the flow reaches \( t = T_0 \), a fact related with the relatively low Reynolds number.

Figure 4 quantifies the flowfield shown in figure 3 in terms of the development of the eccentricity, \( \varepsilon_c \), of one vortex (upper) and its radius (lower) with time. As far as \( \varepsilon_c \) is concerned, one notes that this quantity has two components: the first oscillates with decaying amplitude around a mean state and the second is the linear growth of that mean state with time. A similar linear dependence is expected theoretically\textsuperscript{6} of the square of the radii of the vortices during the relaxation process, in both the counter- and the present co-rotating vortex case. Indeed, this is the behavior shown by the quantity \( a^2 = a_0^2 + 4\nu t \) plotted in the lower part of figure 4. On the other hand, the baseflow period is measured by monitoring flow quantities at two arbitrary field points: the result is shown in figure 5 and confirms the theoretical prediction. This way the turnover time, defined as the time needed by a vortex to return to the same position, is \( 2T_0 \). Finally, figure 6 shows the analogous baseflow results for the full four-vortex system at \( t = 0(T_0/4)T_0 \).

B. Instability Analyses

Turning our attention to Floquet analyses, our first concern has been with the validation of our tools on the well-documented circular cylinder\textsuperscript{4} and the more recently completed analogous study in the wake of the T-106/300 low-pressure turbine case.\textsuperscript{14} Results are shown in figures 7 and 8, the parameters of the latter analysis being \( Re = 895 \) and 2000. In the problem at hand, figure 9 shows the convergence of the leading Floquet multiplier corresponding to three-dimensional (\( \alpha = 1 \), i.e. \( L_x = 1\pi \)) time-periodic perturbations superimposed upon the \( a_0/b_0 = 0.2 \) baseflow discussed earlier. The key observations in this results are, firstly the low level of convergence of the eigenmode and secondly, the large number of iterations necessary to reduce the residual by approximately two orders of magnitude. At these parameters, the Floquet multiplier
is \( \mu = (0.93691, 0.00000) \), corresponding to a stable \((\mu < 1)\) perturbation: the corresponding eigenvector is presented in figure 10. Note that the CPU time for each iteration is proportional to the period of the basic flow, which in turn depends on the distance between the vortices according to (30). The present analyses have been performed serially on a top-end PC; given that each iteration takes approximately 5 CPU hours, the total cost of a serial Floquet computation is of the order of one week, making clear why the low aspect-ratio case has been the first to be considered. In order to complete the analysis of this particular case, runs at different axial wavenumbers at this, relatively inexpensive configurations, are currently underway.

The case \( a_0/b_0 = 0.1 \) has been considered next. The doubling of the initial distance results in four times the computational cost per axial wavenumber, i.e. the results of figure 11 were obtained after about one month of serial computing. The value of \( \alpha = 0 \) was used here and an unstable Floquet multiplier \( \mu = (1.14671, 0.00000) \) has been obtained. In a manner analogous with the case analyzed earlier, this perturbation corresponds to a stationary mode. Figure 11 asserts that three significant digits of \( \mu \) are converged in this case. This means that the basic flow at \( a_0/b_0 = 0.1 \) is unstable to two-dimensional time-periodic perturbations. The amplitude functions of these disturbances are shown in figure 12. Three-dimensional Floquet analyses of this, as well as the full four-vortex configurations, are currently underway.

IV. Conclusions and Outlook

Spectrally-accurate two-dimensional DNS methodologies have been utilized in order to obtain the time-evolution of a pair of co-rotating vortices at different aspect ratios. The periodic orbits established past the initial transients were discretized by a large number of snapshots and analyzed using Floquet theory. From a numerical point of view, what has become clear by the present work is that the computing cost of serial Floquet analyses is prohibitively high for parametric studies to be performed. Nevertheless, converged results for the Floquet multipliers have been obtained in the case of well-separated vortices of equal strengths. The main finding has been that the periodic orbits defined by the two-dimensional basic states, as obtained by the "temporal DNS" approach, are unstable.

Further, two questions of physical origin still remain open at present. First, the effect of the relatively low Reynolds number on the instability of the periodic orbit is not yet analyzed; work is underway in this direction. Second, by contrast to the well-justifiable circular cylinder\(^4\) and low-pressure turbine\(^14\) configurations, axial periodicity is artificially imposed in the present numerical model, as well as in the majority of analogous investigations. The instability of the corresponding periodic orbits may imply that such basic states simply do not exist. Work on answering these questions is currently underway and results will be reported in due course.

Acknowledgments

The material is based upon work partially sponsored by contract numbers AST4-CT-2005-012338 (EU STREP Far-Wake) and TRA2005-08983/TAIR (Ministerio de Educación y Ciencia, Plan Nacional de Investigación). Part of this work has been presented as AIAA Paper 2007-4539.
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Figure 1. Complete picture of the hybrid computational mesh (upper) and detail of the refined center (lower).
Figure 2. Two-vortex system. $Re = 8000, a_0/b_0 = 0.2, \Gamma = \pi, T_0 = 25\pi$. Shown are snapshots of axial vorticity at $t \approx 0, T_0/4, T_0/2$. 
Figure 3. Two-vortex system. $Re = 8000, \alpha_0/\beta_0 = 0.1, \Gamma = \pi, T_0 = 100\pi$ Shown are snapshots of axial vorticity at $t = 0$ (top), $t = T_0/4, T_0/2, 3T_0/2$ (middle), $t = T_0$ (lower).
Figure 4. Evolution of the global eccentricity $\varepsilon_c$ of a pair of Gaussian vortices versus the viscous time scale for $Re = 8000$ and $a_0/b_0 = 0.1$ (upper) and of the square of the vortex radius $a^2$ of one vortex, versus the viscous time scale for $Re = 8000$ and $a_0/b_0 = 0.1$ (lower).
Figure 5. Two-vortex studies, $Re = 8000, a_0/b_0 = 0.1, \Gamma = \pi$ Evolution of the axial velocity component at two arbitrary points in the field, used to compute the period, $T_0 \approx 300\pi$. 

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Figure 6. Four-vortex system. $Re = 8000, a_0/b_0 = 0.1, \Gamma = \pi$ Shown are snapshots of axial vorticity at $t = 0, T_0/2, 3T_0/2, 2T_0$. Shown are snapshots of axial vorticity at $t = 0$ (top), $t = T_0/4, T_0/2, 3T_0/2$ (middle), $t = T_0$ (lower).
Figure 7. Leading Floquet eigenmode (upper) of the periodic baseflow (lower) in the wake of a circular cylinder at $Re = 200$. 

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Figure 8. Vorticity fields of base flow (upper), leading long wavelength eigenmode (lower) in the T-106/300 LPT case at $Re = 895$ (left) and $Re = 2000$ (right).
Figure 9. Convergence history of the most unstable Floquet eigenvalue of the two-vortex system at $Re = 8000$, $\alpha = 1$, $a_0/b_0 = 0.2$. 

The graph shows the residuals decreasing with the number of iterations, indicating a satisfactory convergence of the solution.
Figure 10. Amplitude functions of the leading Floquet mode of the two-vortex system at $Re = 8000, \alpha = 1, a_0/b_0 = 0.2, \Gamma = \pi, T_0 = 25\pi$. Upper left: $\tilde{u}$, Upper right: $\tilde{v}$, Lower left: $\tilde{w}$, Lower right: $\tilde{p}$. 
Figure 11. Convergence of the most unstable eigenvalue for $Re = 8000, \alpha = 0, a_0/b_0 = 0.1$. 
Figure 12. Amplitude functions of the leading Floquet mode of the two-vortex system at $Re = 8000, \alpha = 0, a_0/b_0 = 0.1, \Gamma = \pi, T_0 = 25\pi$. Left: $u$, Middle: $v$, Right: axial disturbance vorticity.