Waves on vortices

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FAR-WAKE project, subtasks 1.1.1 «Vortex meandering »
1.1.2 « End effects »

Thanks to:

P. Meunier, L. Nybelen, J. Fontane, P. Brancher, S. Le Dizès, L. Jacquin, D. Sipp
A. Andersen, T. Bohr, V. de Felice, P. Luchini
Summary

I. Kelvin waves
   A. History
   B. Zoology
   C. Experiments (P. Meunier, IRPHE)

II. Axial flow effects

III. Wave fronts
   A. Linear results
   B. Experiments (P. Meunier, IRPHE)
   C. Simulations (L. Nybelen, CERFACS)

IV. « Bathtub » vortices
   A. Vortex formation
   B. Excitation of waves
   C. Nonlinear energy transfers

FW 1.1.1
« vortex meandering »

FW 1.1.2
« end effects »

Outside FW
« Vibrations of a columnar vortex »
W. Thomson (Lord Kelvin)
Phil. Mag. 10, p. 155, 1880

« A columnar vortex in a perfect fluid can vibrate at specific frequencies »
Expected applications in 1880:
Atomic physics!
W. Thomson (Lord Kelvin)


« Helmholtz’s admirable discoveries of the law of vortex motion [...] inevitably suggests that vortex rings are the only true atoms »

- A speculative theory:
  
  Atoms = Vortices in aether
  Atomic species = Particular knotted vortex structures
  Atomic frequencies = Vibration frequencies of vortices

- A research program:
  
  - Classification of knots
  - Study of vibration modes => 1880 paper

(Precursor of String Theory ????)
A modern application: Aircraft wakes

« I have not the smallest molecule of faith in aerial transport other than ballooning »

Lord Kelvin, 1895.

(In reply to a request to join the royal aeronautical society)
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Linear stability of a vortex: general method

- Base flow: $V_\theta(r)$
- Eigenmodes: $u' = \hat{u}(r)e^{i(kz+m\theta-\omega t)}$
- Axial wavenumber $k$
- Azimuthal wavenumber $m$
- Frequency $\omega = \omega_r + i\omega_i$
  - $\omega_r$: oscillation rate,
  - $\omega_i$: amplification/damping rate

- Dispersion relation: $\omega(m,k)$

Examples:

- $m=0$
- $m=1$
- $m=3$
Vortex models

« Rankine vortex »

\[ V(r) = \begin{cases} \frac{r}{r} & r < 1 \\ \frac{1}{r} & r > 1 \end{cases} \]

- Analytic solution
  (Kelvin, 1880)

- Inviscid theory

- All waves are neutral

« Lamb-Oseen vortex »

\[ V(r) = \frac{1-e^{-r^2}}{r} \]

- Numerical solution
  (Fabre, Sipp & Jacquin 2006; TR 1.1.2-3)

- Viscous theory

- Regular waves + singular modes
Remark: A more realistic vortex model

Two scales model

\[ a_2 \approx 0.1 \, b, \quad a_1 \approx 0.01 \, b \]

Jacquin et al. (2001)

\[ V \sim r^{-1} \]

\[ V \sim r^{-\alpha}, \]
\[ \alpha \approx 1/2 \]

=> Wave dynamics essentially similar to Lamb-Oseen vortex
Vortex models

« Rankine vortex »

\[ V(r) = \begin{cases} \frac{r}{1/r} & r < 1 \\ 1/r & r > 1 \end{cases} \]

- Analytic solution
  (Kelvin, 1880)

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« Lamb-Oseen vortex »

\[ V(r) = \frac{1-e^{-r^2}}{r} \]

- Numerical solution
  (Fabre, Sipp & Jacquin 2006)

- Viscous theory

- Regular waves + singular modes
All waves are regular, Ex = 1000

Outer branches are weakly affected by viscosity
Axisymmetric waves: mechanism
Helical modes (Re = 1000) : Dispersion relation

=> Three main kinds of modes :
Regular « Kelvin » Waves, Singular Modes and Critical Layer Waves
Branch D: the Displacement Wave

Wave involved in: - LW cooperative (Crow) instabilities
- Meandering (receptive to external noise; Fontane et al.)
- Wave fronts (part III)
Displacement wave: Mechanism

\[\text{=> Precession in the direction opposite to vortex rotation}\]
Regular waves.
Example 2: a Steady Wave

(Remark: occurring in short-wave coop. instabilities)
Regular waves.
Example 3: a **Core Wave**

Structure localised within the vortex core
Singular Modes. 
Exemple 1 : a « V » mode

(strongly damped)

(strongly damped)
An example of Critical Layer Wave

- Regular within the core but singular outside
- (weakly) Damped by viscosity but damping rate $\omega_i = O(1)$

Remark: strongly receptive to external perturbations

(Antkowiak & Brancher 04)
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IV. « Bathtub » vortices
   A. Vortex formation
   B. Excitation of waves
   C. Nonlinear energy transfers
Means: Excitation of the Kelvin waves on a vortex created by a rotating flat plate instead of a wing

Advantage: easy experiment, controlled profile of the vortex, laminar vortex, no axial flow

Dye visualisation of the vortex in a cross-cut section

Vorticity measured by PIV
Velocity profiles

- 3 different Reynolds numbers (Re=Γ/ν) ranging from 800 to 14000
- 2 types of excitation: sinusoidal and end effect
- 2 different velocity profiles:

**Gaussian vortex:**

![Gaussian vortex diagram]

**Two-scale vortex (Jacquin et al. 2001):**

![Two-scale vortex diagram]

Temporal evolution of core size

Azimuthal velocity
Excitation of a single wave on an “infinite” vortex

- Excitation by a sinusoidal perturbation of the edge
- Undulation of the vortex with an imposed wavelength
- Visualisation by two laser sheets

Measure the angle $\psi$ for:
- a Gaussian profile
- 3 different Re
- 8 different wavelengths

Temporal evolution:
Dispersion relation

\[ a^2 = a_0^2 + 4\nu t \Rightarrow 2 \text{ parameter fit:} \]
\[ \psi = \psi_0 + C\Gamma/8\pi \nu \ln(1+4\nu t/a_0^2) \]
with \( C \) the dimensionless frequency

Temporal evolution of the angle

Spectrum of the Kelvin waves (m=1)
numerical code (lines) and experiment (symbols)

- good agreement with numerics
- first experimental evidence of a Kelvin mode with a critical layer
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The gaussian trailing vortex model

\[ V_\theta(r) = \frac{\Gamma}{2\pi} \frac{1-e^{-r^2/a^2}}{r} \]

\[ U(r) = U_\infty + \Delta U \ e^{r^2/a^2} \]

- A convenient model
- A simplification of Batchelor’s solution

(see Fernandez Feria et al.)
The gaussian trailing vortex model

\[ V_\theta(r) = \frac{\Gamma}{2\pi r} \left(1 - e^{-r^2/a^2}\right) \]

\[ U(r) = U_\infty + \Delta U \ e^{-r^2/a^2} \]

Nondimensionalisation:

The « q-vortex »
- Length scale : \(a\)
- Time scale : \(a/\Delta U\)

Parameters:
- Swirl number
  \[ q = \frac{\Gamma}{(2\pi a \Delta U)} = 1.57 \frac{V_{\theta\text{max}}}{\Delta U} \]
- Reynolds
  \[ Re_U = a \Delta U / \nu \]
  (a swirling jet model)

The « anti-q-vortex »
- Length scale : \(a\)
- Time scale : \(2\pi a^2/\Gamma\)

Parameters:
- Axial flow parameter (Rossby)
  \[ \varepsilon = \frac{2\pi a \Delta U}{\Gamma} = q^{-1} \]
- Reynolds
  \[ Re_\Gamma = \frac{\Gamma}{(2\pi \nu)} = q \ Re_U \]
  (a trailing vortex model)
The Gaussian trailing vortex model: Stability properties

Four families of instabilities:

1. «Centrifugal-like» (helical) instabilities
   \[ q < 1.5 \quad \text{(or } \epsilon > 0.67 \text{)}, \ m < 0 \]
   Strong

2. «Inviscid centre modes»
   *Heaton (2007)*
   \[ 1.5 < q < 2.31 \quad \text{(or } 0.433 < \epsilon < 0.67 \text{)}, \ m = -1 \]
   extremely weak

3. «Khorrami modes»
   *Khorrami (1991)*
   \[ q < 1.3 \quad \text{(or } \epsilon > 0.77 \text{)}, \ m = 0 \text{ and } 1. \]
   extremely weak

4. Viscous centre modes
   *Fabre & Jacquin (2004); Le Dizès & Fabre (2007); Fabre & Le Dizès (2008)*
   \[ q < 0.141 \ Re_U^{1/3} \quad \text{(or } \epsilon > 4.35 \ Re_T^{-1/4} \text{)}, \ m < 0 \]
   Weak but significant
The gaussian trailing vortex model: Stability properties

Four families of instabilities:

1. «Centrifugal-like» (helical) instabilities
   Leibovich & Stewartson (1983), Mayer & Powell (1992)
   \( q < 1.5 \) (or \( \varepsilon > 0.67 \)), \( m < 0 \)
   Strong

2. «Inviscid centre modes»
   Heaton (2007)
   \( 1.5 < q < 2.31 \) (or \( 0.433 < \varepsilon < 0.67 \)), \( m = -1 \)
   extremely weak

3. «Khorrami modes»
   Khorrami (1991)
   \( q < 1.3 \) (or \( \varepsilon > 0.77 \)), \( m = 0 \) and \( 1 \).
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4. Viscous centre modes
   Fabre & Jacquin (2004); Le Dizès & Fabre (2007); …
   \( q < 0.141 \ Re^{1/3} \) (or \( \varepsilon > 4.35 \ Re^{-1/4} \)), \( m = -1 \)
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Wave fronts

A wave front (or wavepackest) is a localised structure propagating along the vortex core. (analogy: shock wave, hydraulic jump)

WF are generated by any local event:
- Walls of experimental facilities (end effects, subtask 1.1.2)
- Aircraft maneuveral (acceleration, turning, etc…)
- Atmospheric events (turbulence, etc…)
- Vortex linking (Crow instability)

Focus:
A. Linear results (UPS-IMFT)
B. Experiments (CNRS-IRPHE)
C. Numerical simulations (CERFACS)
(other simulations by UCL and ONERA)
Input from linear theory:
group velocity \( c_g = \frac{\partial \omega}{\partial k} \)

\[ m = 0 \]
\[ |c_g| = 0.63 \approx V_{\theta max} \]
\[ ka << 1 (\lambda >> a) \]
=> Axisymmetric
(or pressure) front

\[ m = 1 \]
\[ |c_g| = 0.29 \approx 0.45 V_{\theta max} \]
\[ ka \approx 0.45 (\lambda \approx 10 a), \text{ branch D} \]
=> Helical (or corkscrew) front
A. Linear response to localised perturbations

=> 3 cases considered:

- Axisymmetric « pinch » (m=0)
- Hélicoal « twist » (m=1)
- « Twisted flattening » (m=2)

Method: projection on linear eigenmodes (Fabre, 2002)
Réponse à une condition initiale :
« pincement » (Re = 1000)
Condition initiale : « pincement »
(Re = 1000)
Condition initiale : « vrille »
(Re = 1000)
Initial value problem for a Lamb-Oseen vortex (Re = 1000)

Initial condition = «helical twist»
Réponse à une cond. init. : détail

« corkcrew wave front » (ondes de K.)
Filamentation (singular modes)
Corotating wave front (ondes de K.)
Condition initiale :
« aplatissement torsadé »
Condition initiale :
« aplatissement torsadé » (Re = 1000)
« Aplatissement torsadé » : détail
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Experiments in water tank
P. Meunier, F. Strina, T. Leweke,
CNRS-IRPHE (TR 1.1.2-1)

Means: Excitation of the Kelvin waves on a vortex created by a rotating flat plate instead of a wing

Advantage: easy experiment, controlled profile of the vortex, laminar vortex, no axial flow

Dye visualisation of the vortex in a cross-cut section

Vorticity measured by PIV
End effects: dye visualisations

- creation of a disrupted vortex at the edge of the rectangular plate
- axial velocity induced by an axial variation of circulation
- appearance of helical waves
End effects: axial velocity

Axial position of dye patches (Re=3960)

Dimensionless velocity of the fluid and the shock

2 different velocities:
- \( u_z \) = velocity of the fluid (slope of the experimental symbols)
- \( u_{sh} \) = velocity of the “shock” propagating along the axis (black line)
End effects: helical waves

- Measure of the mean wavelength
- Measure of 2 different velocities:
  
  \[ u_{ph} = \text{phase velocity (slope of the experimental symbols)} \]
  
  \[ u_{sh} = \text{group velocity (black line)} \]
End effects: dispersion relation

- Phase and group velocities independent of Reynolds number
- Phase velocity in agreement with theoretical dispersion relation (with axial velocity)
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C. Numerical simulation

(L. Nybelen, CERFACS)

1. Single axisymmetric front

Amplitude = ratio of radii $r_{c2}/r_{c1}$
Global dynamics
• « Pressure » wave with nearly constant velocity
• Annular vorticity structure propagates along vortex core
• Génération of axial flow in vortex core

\[
\frac{\partial \omega_\theta}{\partial t} = \omega_z \frac{\partial \nu_\theta}{\partial z}
\]

**DNS configuration** \( rc_2/rc_1 = 2 \)  \( Re = 10^4 \)

\[ t/T_{rot} = 2.29 \]

\[ t/T_{rot} = 6.27 \]
Velocity of front vs amplitude
=> Good agreement with theory

Axial flow behind front measured by Swirl parameter $q = 1.12 \alpha \frac{v_{\theta max}}{w_{max}}$

=> unstable to inviscid «centrifugal-like» modes
Subsequent dynamics for $r_{c_2}/r_{c_1}=2$, $Re_\Gamma=10^4$, with initial noise

Development of inviscid instabilities

=> « Vortex bursting of kind I »
C. Numerical simulation
(L. Nybelen, CERFACS)

2. Collision of two fronts

Modelisation: two fronts on a rectilinear vortex

Amplitude parameter: $\varepsilon = \frac{|r_{c2}-r_{c1}|}{r_{c1}}$
Global dynamics

Simulation DNS (ε=1 et ReΓ=10^4)
Dynamique illustrée par deux iso surfaces de vorticité

• Propagation de deux fronts d’ondes, caractérisés par une structure annulaire de vorticité

• Collision des fronts: changement soudain de la taille du tourbillon

• Perturbations déclenchant le développement d’instabilités hélicoïdales
  • dynamique complexe dans la région de collision
• Perte structurelle du tourbillon dans la région de collision et début de propagation d’ondes dans le sens inverse

\[ t^* = 9.58 \]

• Extension de la région déstructurée et développement d’instabilités hélicoïdales

\[ t^* = 11.14 \]

=> « Vortex bursting of kind II »

Criterion : \( \varepsilon = \frac{|r_{c2} - r_{c1}|}{r_{c1}} > 0.4 \)
Conclusions on wave fronts

- Linear theory predicts two kinds of fronts:
  - Axisymmetric (« pressure front ») \( c_g \approx V_{\theta_{\text{max}}} \)
  - Helical displacement (« corkscrew front ») \( c_g \approx 0.45V_{\theta_{\text{max}}} \)

- Both are observed in experiments

- Numerical simulations show that pressure fronts lead to two kinds of vortex burstings
  - « kind I » : single front of large amplitude
  - « kind II » : collision of two fronts.
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The “bathtub vortex”

« Water sculpture » by William Pye
The “bathtub vortex”

Motivations:
- A spectacular flow with few fundamental studies
- Application:
  - Hydroelectric plants
- A « minimalistic » vortex model for fundamental studies

Three ongoing studies:

A. Mechanism of vortex creation
   (with V. De Felice & P. Luchini)

B. Excitation of waves
   (with A. Andersen & T. Bohr)

C. Nonlinear energy transfers
   (with J. Fontane)
A. Mechanism of vortex creation in a non-rotating environment

with V. De Felice & P. Luchini, Università di Salerno, Italy

Linear stability (+ experiment) of drain flow

Axisymmetric setting:
No instability leading to rotation

Non-axisymmetric setting:
Instability above critical flow rate
=> Vortex appears through supercritical bifurcation
(cf. Kawakubo et al., 1975)
Most unstable linear eigenmode:

Iso-levels of velocity modulus

=> Onset of rotation (not yet a vortex)
B. Excitation of waves
with A. Andersen & T. Bohr, DTU, Copenhagen

Strongly rotating tank with
draining hole at bottom
and continuous alimentation
At periphery
B. Excitation of waves
with A. Andersen & T. Bohr, DTU, Copenhagen
A simple model: the “hollow core vortex”

- Potential vortex (Circulation $\Gamma$)
- No axial flow
- Cylindrical interface (Radius $a_0$)
- Parameters: $Re = \frac{\Gamma}{2\pi v}$; $We = \frac{\sigma a_0}{\rho} \left( \frac{2\pi}{\Gamma} \right)^2$

Stability properties:

- Analytical solution
  Ponstein (1959)… and yet Kelvin (1880)!!
- Two waves for each (m,k)
- Remark: instability (plateau-like) for $We>0$
A possible mechanism: Multipolar instability due to strain field caused by rod

Resonant cases (for $We=0$, $Re=\infty$)
Shape of the vortex interface for some unstable configurations

\[(m_a, m_b) = (0, 2)\]
\[(m_a, m_b) = (0, 3)\]
\[(m_a, m_b) = (-2, 2)\]

Andersen & Bohr
C. Nonlinear energy transfers
with J. Fontane

**Objective**: study long-time behaviour of *single vortices*

- Stability of monochromatic wavetrain (*Benjamin-Feir instab.?*)
- Energy cascade ? (direct/inverse ?)
- Solitary waves ?

*The Hollow Core Vortex allows to adress these issues using semi-analytical methods*
C. Nonlinear energy transfers
with J. Fontane

**Situation:**

**Vortex interface**  \( a(\theta, z, t) = a_0 + \sum_{m,k} \eta_{m,k}(t) e^{i(m\theta + kz)} \)

**Velocity potential**  \( \Phi(r, \theta, z, t) = \Phi_0 + \sum_{m,k} \phi_{m,k}(t) \frac{K_m(kr)}{K_m(ka_0)} e^{i(m\theta + kz)} \)

\[=> \text{Nonlinear system:} \quad \frac{d\eta_{m,k}(t)}{dt} = \ldots \]

\[\frac{d\phi_{m,k}(t)}{dt} = \ldots \]

(difficulty: third-order nonlinearities)
C. Energy transfer in wave spectra
with J. Fontane

First results :
(axisymmetric case)

A. « localised pinch »
(cf. part III)

B. Large amplitude wavetrain
Thank you!