SPATIAL STABILITY OF SPECIFIC VORTICES

Spatial stability of q-vortex and Batchelor vortex for high swirl numbers

Viscous modes for large swirl numbers
Convective and absolute instabilities
Parallel and non-parallel spatial analyses
Vortex meandering assessment

By

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Our participation in the FAR-WAKE Project:
Task 1.1: Waves on vortex
Subtask 1.1.1: Vortex meandering
Previous and simultaneous work


- Viscous modes for high swirl numbers
  - Fabre and Jacquin (2004). JFM 500, 239

- Asymptotic stability analyses

- Spatial stability analyses
Spatial stability

• Provides directly physical relevant information such as the frequency ranges of the unstable modes in terms of the other parameters of the flow (Reynolds number and the swirl parameter), and in terms of the azimuthal wave number of the perturbations.

• Onset of absolute instability in a quite straightforward way [Parras and Fernandez-Feria (2007), JFM 583, 27].

• Is the only appropriate method to account for non-parallel effects based on the Parabolized Stability Equations [del Pino, Parras and Fernandez-Feria (2008), submitted to Phys. Fluids].
Spatial stability q-vortex

Velocity field
\[ U = 0 \quad V = \frac{q}{r} \left(1 - e^{-r^2}\right) \quad W = W_0 + e^{-r^2} \]

Dimensionless parameters
\[ S_w = \frac{V_c}{W_c} \quad q \equiv S_w \quad Re = \frac{r_c W_c}{\nu} \]
Spatial stability q-vortex

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Perturbations decomposed as normal modes

\[ s \equiv [u', v', w', p']^T = S e^{az + i(n\theta - \omega t)} \]

The eigenvalue and the eigenfunction

\[ a \equiv \gamma + i\alpha \quad S(r) \equiv \begin{pmatrix} iF(r) \\ G(r) \\ H(r) \\ \Pi(r) \end{pmatrix} \]

We look for viscous centre modes for large swirl numbers (q >> 1).
Convective instabilities $q$-vortex

**Results**

Comparison with the asymptotic results by Le Dizès and Fabre (2007)
Absolute instabilities q-vortex

\[ Re=10^3, W_0=-1, n=-1; \omega_i = -1.6151 \]

\[ Re=10^3, W_0 = -0.75 \]

\[ q=1.78, \omega_i = 0 \]
\[ q=1.775, \omega_i = 0 \]
\[ q=1.77, \omega_i = 0 \]
\[ q=1.775, \omega_i = 0.01 \]
\[ q=1.775, \omega_i = 0.001 \]

\[ Wo < 0 \]
Absolute instabilities q-vortex

\[ n = -1 \]

\[ \text{Re}=10^6 \quad \text{Stable} \]

\[ q_{ca} \]

\[ -1.5 \quad -1 \quad -0.5 \]

\[ \omega_0 \]

\[ -1.5 \quad -1 \quad -0.5 \]
Absolute instabilities q-vortex

\[ q_{ca,\text{max}} \sim c_n R e^{1/3} \]

\[ c_1 \approx 0.1408 \quad c_2 \approx 0.1142 \quad c_3 \approx 0.0858 \]

(Fabre and Le Dizès, 2008)
Velocity field of the non-parallel Batchelor Vortex (BV) (1964)

\[ U(r, z) = 0 \quad V(r, z) = \frac{q_1}{r} \left( 1 - e^{-r^2/z} \right) \]

\[ W(r, z) = 1 + \frac{q_1^2}{2z} Q\left(\frac{r^2}{z}\right) - \left[ q_1^2 \ln \left( \frac{Re_1^2 z}{4} \right) + \frac{Re_1}{4\delta} \right] \frac{e^{-r^2/z}}{2z} \]

\[ Q(\eta) \equiv e^{-\eta}[\ln \eta + E_1(\eta) - 0.807] + 2E_1(\eta) - 2E_1(2\eta) \]

\[ E_1(\eta) \equiv \int_{\eta}^{\infty} dx \frac{e^{-x}}{x} \]

**q-vortex:**

\[ U = 0 \quad V = \frac{q}{r} \left( 1 - e^{-r^2} \right) \quad W = W_0 + e^{-r^2} \]
Parameters:

\[ r_c \equiv \sqrt{\frac{4\nu z_c}{W_c}} \]
\[ Re_1 = \frac{W_c r_c}{\nu} \]
\[ \Delta \equiv \frac{r_c}{z_c} = \frac{4}{Re_1} \]
\[ W_c \equiv W_\infty \]
\[ q_1 = \frac{\Gamma_0}{2\pi r_c W_c} \equiv \frac{V_c}{W_c} \]
\[ \delta \equiv \frac{z_c r_c}{L} \]

Comparison with the parallel \( q \)-vortex: \( W_{cP} = W_c F(z) \)

\[ F(z) \equiv \frac{q_1^2}{2} \ln \left( \frac{Re_1^2 z}{4} \right) + \frac{Re_1}{8\delta} \quad W_0 \approx -\frac{z}{F(z)} \]

that yields

\[ q = \frac{q_1}{F(z)} \]
\[ Re = Re_1 F(z) \]
Comparison between the parallel q-vortex and the BV: W and dW/dr

$Re=1000$, $W0=-1$ and $z=1$ for 3 different $q$:

$q=0$ (a), $q=1$ (b) and $q=1.78$ (c).

$F(z)=1$: $Re_1 = Re$ and $q_1 = q$

Batchelor-vortex (W – , dW/dr . -)

q-vortex (W --, dW/dr .)
The spatial stability is studied with two different approximations:

**Near-Parallel (NP) approx.**
(q-vortex, depends on $r$)

$$\alpha \equiv \gamma + i\alpha$$

$$s \equiv [u', v', w', p']^T = S e^{az + i(n\theta - \omega t)}$$

**Non-parallel PSE**
(Batchelor vortex, depends on $r,z$)

$$s(r, \theta, z, t) = S(r, z) \chi(z, \theta, t)$$

$$\chi(z, \theta, t) = \exp \left[ \frac{1}{\Delta} \int_{z_0}^{z} a(z')dz' + in\theta - i\omega t \right]$$

$$S(r) \equiv \begin{pmatrix} iF(r) \\ G(r) \\ H(r) \\ \Pi(r) \end{pmatrix}$$

$$S(r, z) \equiv \begin{pmatrix} iF(r, z) \\ G(r, z) \\ H(r, z) \\ \Pi(r, z) \end{pmatrix}$$

**Parabolized Stability Equations (PSE) features:**
1) Take into account the ‘history’ of the disturbances.
2) Normalization condition
\[ L \cdot S + \Delta \, M \cdot \frac{\partial S}{\partial z} = 0 \quad \text{being the non-dimensional parameter} \quad \Delta \equiv \frac{r_c}{z_c} \]

\[ L \equiv L_1 + aL_2 + \frac{1}{Re}L_{31} + \Delta L_{32} + a^2 \frac{1}{Re}L_4 \]

\[
L_1 = \begin{pmatrix}
\frac{1}{r} + \frac{\partial}{\partial r} & i \left( \frac{nV}{r} - \omega \right) & \frac{2V}{r} & 0 & 0 \\
\frac{rV}{\partial r} + \frac{V}{r} & i \left( \frac{nV}{r} - \omega \right) & 0 & 0 & \frac{\partial}{\partial r} \\
0 & 0 & i \left( \frac{nV}{r} - \omega \right) & 0 & \frac{m}{r}
\end{pmatrix}
\]

\[
L_2 = M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
W & 0 & 0 & 0 \\
0 & W & 0 & 0 \\
0 & 0 & W & 1
\end{pmatrix}
\]

\[
L_{31} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-D_r^2 + \frac{n^2+1}{r^2} & 0 & 0 & 0 & 0 \\
-\frac{2in}{r^2} & -D_r^2 + \frac{n^2+1}{r^2} & 0 & 0 & 0 \\
0 & 0 & -D_r^2 + \frac{n^2+1}{r^2} & 0 & 0 \\
0 & 0 & 0 & -D_r^2 + \frac{n^2+1}{r^2} & 0
\end{pmatrix}
\]

\[
L_{32} = \begin{pmatrix}
\frac{\partial U}{\partial r} & 0 & 0 & 0 & 0 \\
U \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) & 0 & 0 & 0 & 0 \\
0 & U \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) & 0 & 0 & 0 \\
0 & U \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) & 0 & 0 & 0 \\
0 & U \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) & 0 & 0 & 0
\end{pmatrix}
\]

\[
L_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]
Non-parallel stability of BV

Parameters in the Batchelor vortex

\[ r_c = \epsilon b \quad z_c = \left( \frac{\epsilon}{2} \right)^2 b \text{Re}_b \]

\[ \text{Re}_b \equiv \frac{W_\infty b}{\nu} \quad \text{Re}_1 = \epsilon \text{Re}_b \quad \frac{D}{\rho} \simeq \frac{1}{2} \pi L W_\infty^2 \]

\[ D = C_D \rho W_\infty^2 S/2 \quad C_D \simeq C_L^2 / (\pi A_R \epsilon) \quad \epsilon = 1 \]

\[ \delta \equiv \frac{z_c r_c}{L} \quad \delta \simeq \left( \frac{\pi A_R}{2 C_L} \right)^2 \epsilon^3 \text{Re}_b = \left( \frac{\pi A_R}{2 C_L} \right)^2 \epsilon^2 \text{Re}_1 \quad L \simeq \left( \frac{C_L b}{\pi A_R} \right)^2 \]

\[ C_L = 0.7 \]

\[ A_R \equiv b^2 / S = 10 \]

\[ \text{Re}_1 \quad q_1 \]

\[ \Gamma_0 \simeq \frac{C_L W_\infty b}{2 s_o A_R} \quad s_o = \pi / 4 \quad \text{Re}_\Gamma \equiv \frac{\Gamma_0}{\nu} \simeq \frac{2 C_L}{\pi A_R} \text{Re}_b \]
Case $\text{Re}_1 = 10^5$, $z = 1$ and different $q_1$

$\gamma = -1$

$\gamma_{\text{max}}$

Numerics (NP)


$$\omega \sim n q_1 + \frac{n(1 - F)}{2 q_1} \left[ 0.192 q_1^2 + F \pm \sqrt{(0.192 q_1^2 + F)^2 + \frac{4 k_n}{n} \text{Re}^{-1/2} q_1^{3/2}} \right]$$
Non-parallel stability of BV

Case $Re_1 = 10^5$, $q_1 = 0.0325$ and $z = 1$

$\gamma_{\text{max}} (\omega = -0.31)$

$a \equiv \gamma + i\alpha$
Case $\text{Re}_1 = 10^5$ and $q_1 = 0.0325$ increasing $z$

\[ a = \gamma + i\alpha \]

\[ \omega = -0.31 \]

\[ Z = \frac{\tilde{z}}{r_c} = \frac{z\text{Re}_1}{4} \]
Case $Re_1 = 10^5$ and $q_1 = 0.0325$ increasing $z$

Eigenfunctions

$Z = 500$

$Z = 5000$

$Z = 11325$
Case $Re_1 = 10^5$ and $q_1 = 0.06$ increases $z$

$n = -1$

$z = 1$

$\omega = -0.6$

$Z = \frac{\bar{z}}{r_c} = \frac{zRe_1}{4}$

$a \equiv \gamma + i\alpha$
Case $\text{Re}_1 = 10^5$, $q_1 = 0.06$ increasing $z$

Eigenfunctions $Z = 3125$

PSE predicts a much shorter stabilization distance than NP, but in any case it is much further downstream than the appearance of vortex meandering.
Stability of the vortex velocity profiles measured experimentally for which meandering has been observed (Devenport et al. (1996), Roy & Leweke (2005))

Gaussian ($q$-vortex) fitting

$$
\tilde{V} = V_{\text{max}} \left( 1 + \frac{1}{2\sigma} \right) \frac{a}{\tilde{r}} \left[ 1 - e^{-\sigma \tilde{r}^2 / a^2} \right] \quad \sigma \simeq 1.25643
$$

$$
\tilde{W}(\tilde{r} = 0) = W_{\infty} - W_D
$$

$$
r_c = \frac{a}{\sqrt{\sigma}} \quad q \quad \text{or} \quad q_1 = \frac{V_{\text{max}}[1 + 1/(2\sigma)]\sqrt{\sigma}}{W_c}
$$

$$
Re_c = \frac{W_{\infty} c}{\nu}
$$

$q$-vortex parameters:

$$
q \simeq 1.567 \frac{V_{\text{max}}}{W_D} \quad Re = 0.8921 \frac{Re_c a W_D}{c W_{\infty}} \quad W_0 = -\frac{W_{\infty}}{W_D}
$$

$BV$ parameters:

$$
q_1 \simeq 1.567 \frac{V_{\text{max}}}{W_{\infty}} \quad Re_1 = 0.8921 \frac{Re_c a}{c}
$$
Several experiments (Devenport et al. JFM 312:67-106, 1996)

\[ \text{Re}_c = 5.3 \times 10^5 \]

<table>
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<th>No</th>
<th>( \tilde{z}/c )</th>
<th>( a/c )</th>
<th>( W_D/W_\infty )</th>
<th>( V_{max}/W_\infty )</th>
<th>( \text{Re} )</th>
<th>( q )</th>
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<td>0.166</td>
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<td>1.81</td>
<td>-6.94</td>
<td>8948</td>
<td>0.260</td>
</tr>
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</table>

Experiments by Roy and Leweke (Int. Conference on High Reynolds numbers Vortex Interactions, Toulouse, 2005)

\[ \text{Re}_c = 7.5 \times 10^5 \]

<table>
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<tr>
<th>No</th>
<th>( \tilde{z}/c )</th>
<th>( a/c )</th>
<th>( W_D/W_\infty )</th>
<th>( V_{max}/W_\infty )</th>
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<td>7135</td>
<td>1.33</td>
<td>-6.67</td>
<td>47570</td>
<td>0.313</td>
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Most unstable viscous mode \((n = -1)\)

\[
w = 1.121 |\omega| \frac{c}{\alpha} \frac{W_D}{W_\infty} \quad \lambda = \frac{5.605 a}{\alpha} \frac{c}{W}
\]

| \(N^\circ\) | \(\alpha\) | \(|\omega|\) | \(\lambda\) | \(w\) |
|---|---|---|---|---|
| 4 | 0.309 | 3.50 | 0.35 | 29.7 |
| 5 | 0.256 | 2.9 | 0.72 | 21.7 |
| 6 | 0.542 | 4.29 | 0.73 | 10.2 |

The frequency reported experimentally by Devenport et al. is less than unity!!

Recent (2008) experimental results by Roy and Leweke (FAR-Wake TR111-4, presented later in this workshop) find frequencies of meandering of about 1 Hz (case 6), corresponding to

\[
w \approx 0.14
\]

But the azimuthal velocity profile does not fit a Gaussian q-vortex.
1) We fully characterize the viscous stability of $q$-vortex for large swirl numbers and large Reynolds numbers. In particular, the onset of absolute instabilities for azimuthal wavenumbers $n=-1,-2,-3$. A comparison with the asymptotics results by Le Dizès and Fabre is given, showing a very good agreement for high Re.

2) We also consider non-parallel effects in the Bartchelor vortex (BV, whose parallel flow approximation is the $q$-vortex) through PSE. The results differ significantly for very large swirl, but for practical cases of interest this is not the case. However, PSE predicts that the viscous unstable modes becomes stable in a much shorter axial distance than in the parallel $q$-vortex case.

3) We finally analyse the viscous stability properties of the $q$-vortex fitting experimental results by Devenport et al (1996) and by Roy & Leweke (2005) where vortex meandering has been observed. Viscous stability results predict frequencies more than one order of magnitude larger than the experimental ones. For this reason, we suggest that the vortex meandering can not be explained by viscous instabilities. But to be conclusive, the satbility of more realistic wing-tip vortex velocity profiles should be explored.
Parras, L., Fernandez-Feria, R.
Spatial stability and the onset of absolute instability of Batchelor vortex for high swirl numbers
58th Annual Meeting of the Division of Fluid Dynamics - American Physical Society, Chicago (IL.), USA

del Pino, C., Parras, L., Fernandez-Feria, R.
Non-parallel spatial stability of Batchelor vortex
59th Annual Meeting of the Division of Fluid Dynamics - American Physical Society, Tampa Bay (FL), USA

Parras, L., Fernandez-Feria, R.
Spatial stability and the onset of absolute instability of Batchelor vortex for high swirl numbers

del Pino, C., Parras, L., Fernandez-Feria, R.
Non-parallel spatial stability of Batchelor vortex
Physics of Fluids (submitted, 2008).